CHAPTER 10

DERIVATIVES OF COMPOSITE FUNCTIONS

Partial derivatives of a composite function

Composite function of two variables

Let us first consider functions of two variables.

Suppose that a function $f\colon \mathrm{D}_f \subset \mathbb{R}^2 \to \mathbb{R}^1$ of variables (u, v) and a vector function $\boldsymbol{g} = (g_1, g_2)\colon \mathrm{D}_{\boldsymbol{g}} \subset \mathbb{R}^2 \to \mathbb{R}^2$ of variables (x, y) given such that $\mathrm{H}_{\boldsymbol{g}} \subset \mathrm{D}_f$, i.e., $\boldsymbol{g}(x, y) \in \mathrm{D}_f$ for each $(x, y) \in \mathrm{D}_{\boldsymbol{g}}$. Suppose that functions g_1, g_2 have continuous partial derivatives at the point $\boldsymbol{a} \in \mathrm{D}_{\boldsymbol{g}}$ and the function f has continuous partial derivatives at the point $\boldsymbol{b} = \boldsymbol{g}(\boldsymbol{a})$. Then the composite function $h = f \circ \boldsymbol{g}$ has continuous partial derivatives at $\boldsymbol{a}$ and the following relations hold.

$$
\begin{aligned}
\frac{\partial h}{\partial x}(\boldsymbol{a}) &= \frac{\partial f}{\partial u}(\boldsymbol{b})\frac{\partial g_1}{\partial x}(\boldsymbol{a}) + \frac{\partial f}{\partial v}(\boldsymbol{b})\frac{\partial g_2}{\partial x}(\boldsymbol{a}),\\
\frac{\partial h}{\partial y}(\boldsymbol{a}) &= \frac{\partial f}{\partial u}(\boldsymbol{b})\frac{\partial g_1}{\partial y}(\boldsymbol{a}) + \frac{\partial f}{\partial v}(\boldsymbol{b})\frac{\partial g_2}{\partial y}(\boldsymbol{a}).
\end{aligned}
\tag{10.1}
$$

To understand the formula, consider the differentials. For the function $f(u,v)$ we have

$$\mathrm{d}f(\boldsymbol{b},(\mathrm{d}u,\mathrm{d}v)) = \frac{\partial f}{\partial u}(\boldsymbol{b})\mathrm{d}u + \frac{\partial f}{\partial v}(\boldsymbol{b})\mathrm{d}v.$$

Now let

$$u = g_1(x,y), v = g_2(x,y), \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{g}(\boldsymbol{a}) = (g_1(\boldsymbol{a}), g_2(\boldsymbol{a})).$$

Then

$$\mathrm{d}u = \frac{\partial g_1}{\partial x}(\boldsymbol{a})\mathrm{d}x + \frac{\partial g_1}{\partial y}(\boldsymbol{a})\mathrm{d}y\,, \qquad \mathrm{d}v = \frac{\partial g_2}{\partial x}(\boldsymbol{a})\mathrm{d}x + \frac{\partial g_2}{\partial y}(\boldsymbol{a})\mathrm{d}y\,.$$

The differential of the composite function $h(x,y) = f(g_1(x,y), g_2(x,y)$ can therefore be written in the form

$$\mathrm{d}f(\boldsymbol{b},(\mathrm{d}u,\mathrm{d}v)) =$$

$$= \frac{\partial f}{\partial u}(\boldsymbol{b})\left(\frac{\partial g_1}{\partial x}(\boldsymbol{a})\mathrm{d}x + \frac{\partial g_1}{\partial y}(\boldsymbol{a})\mathrm{d}y\right) + \frac{\partial f}{\partial v}(\boldsymbol{b})\left(\frac{\partial g_2}{\partial x}(\boldsymbol{a})\mathrm{d}x + \frac{\partial g_2}{\partial y}(\boldsymbol{a})\mathrm{d}y\right).$$

Putting the terms containing $\mathrm{d}x$ and $\mathrm{d}y$ together, we get

$$\mathrm{d}f(\boldsymbol{b},(\mathrm{d}u,\mathrm{d}v)) = \left(\frac{\partial f}{\partial u}(\boldsymbol{b})\frac{\partial g_1}{\partial x}(\boldsymbol{a}) + \frac{\partial f}{\partial v}(\boldsymbol{b})\frac{\partial g_2}{\partial x}(\boldsymbol{a})\right)\mathrm{d}x + \\ + \left(\frac{\partial f}{\partial u}(\boldsymbol{b})\frac{\partial g_1}{\partial y}(\boldsymbol{a}) + \frac{\partial f}{\partial v}(\boldsymbol{b})\frac{\partial g_2}{\partial y}(\boldsymbol{a})\right)\mathrm{d}y\,.$$

The same differential can also be expressed as

$$\mathrm{d}h(\boldsymbol{a},(\mathrm{d}x,\mathrm{d}y)) = \frac{\partial h}{\partial x}(\boldsymbol{a})\mathrm{d}x + \frac{\partial h}{\partial y}(\boldsymbol{a})\mathrm{d}y\,.$$

Comparing the terms containing $\mathrm{d}x$ and $\mathrm{d}y$ we obtain (11.1):

$$\begin{aligned} \frac{\partial h}{\partial x}(\boldsymbol{a}) &= \frac{\partial f}{\partial u}(\boldsymbol{b})\frac{\partial g_1}{\partial x}(\boldsymbol{a}) + \frac{\partial f}{\partial v}(\boldsymbol{b})\frac{\partial g_2}{\partial x}(\boldsymbol{a}) \\ \frac{\partial h}{\partial y}(\boldsymbol{a}) &= \frac{\partial f}{\partial u}(\boldsymbol{b})\frac{\partial g_1}{\partial y}(\boldsymbol{a}) + \frac{\partial f}{\partial v}(\boldsymbol{b})\frac{\partial g_2}{\partial y}(\boldsymbol{a}) \end{aligned}$$

If any point $\mathbf{a} \in \mathrm{D}_{\mathbf{g}} = \mathrm{D}_h$ can be put to the equalities (11.1), we can consider functions

$$\frac{\partial h}{\partial x_1} = \frac{\partial f}{\partial y_1}\frac{\partial g_1}{\partial x_1} + \frac{\partial f}{\partial y_2}\frac{\partial g_2}{\partial x_1}, \qquad \frac{\partial h}{\partial x_2} = \frac{\partial f}{\partial y_1}\frac{\partial g_1}{\partial x_2} + \frac{\partial f}{\partial y_2}\frac{\partial g_2}{\partial x_2}. \qquad (10.2)$$

Most of the functions that we have investigated until now were composite and we had no problems with the calculation of partial derivatives. The mentioned formulas are necessary in the situations where some of the functions f or g are unknown or represent the whole set of functions.

☛ ***Example 1.*** Find the differential of a composite function

$$h(x,y) = f(u,v), \text{ where } u = g_1(x,y) = e^{xy}, v = g_2(x,y) = e^{-xy}.$$

Solution.

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial g_1}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial g_2}{\partial x} = \frac{\partial f}{\partial u} y e^{xy} + \frac{\partial f}{\partial v}(-y)e^{-xy},$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial g_1}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial g_2}{\partial y} = \frac{\partial f}{\partial u} x e^{xy} + \frac{\partial f}{\partial v}(-x)e^{-xy}$$

Thus

$$\mathrm{d}F((x,y),(h_1,h_2)) = (y e^{xy}\frac{\partial f}{\partial u} - y e^{-xy}\frac{\partial f}{\partial v})h_1 + (x e^{xy}\frac{\partial f}{\partial u} - x e^{-xy}\frac{\partial f}{\partial v})h_2$$

Composite function of n variables

Suppose that a function

$$f\colon \mathsf{D}_f \subset \mathbb{R}^k \to \mathbb{R}^1$$

of variables $(y_1, y_2, \ldots, y_k)$ and a vector function

$$\boldsymbol{g} = (g_1, g_2, \ldots, g_k)\colon \mathsf{D}_{\boldsymbol{g}} \subset \mathbb{R}^n \to \mathbb{R}^k$$

of variables $(x_1, x_2, \ldots, x_n)$ are given such that $\mathsf{H}_{\boldsymbol{g}} \subset \mathsf{D}_f$, i.e., $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}) \in \mathsf{D}_f$ for all $\boldsymbol{x} \in \mathsf{D}_{\boldsymbol{g}}$. Suppose that functions $g_1, g_2, \ldots, g_k$ have continuous partial derivatives at a point $\mathbf{a} \in \mathsf{D}_{\mathbf{g}}$ and the funtion f has continuous partial derivatives at a point $\mathbf{b} = \mathbf{g}(\mathbf{a})$. Then the composite function $h = f \circ \mathbf{g}$ has continuous partial derivatives at $\mathbf{a}$ and the following equalities hold:

$$
\begin{aligned}
\frac{\partial h}{\partial x_1}(\mathbf{a}) &= \frac{\partial f}{\partial y_1}(\mathbf{b})\frac{\partial g_1}{\partial x_1}(\mathbf{a}) + \frac{\partial f}{\partial y_2}(\mathbf{b})\frac{\partial g_2}{\partial x_1}(\mathbf{a}) + \ldots + \frac{\partial f}{\partial y_k}(\mathbf{b})\frac{\partial g_k}{\partial x_1}(\mathbf{a}), \\
\frac{\partial h}{\partial x_2}(\mathbf{a}) &= \frac{\partial f}{\partial y_1}(\mathbf{b})\frac{\partial g_1}{\partial x_2}(\mathbf{a}) + \frac{\partial f}{\partial y_2}(\mathbf{b})\frac{\partial g_2}{\partial x_2}(\mathbf{a}) + \cdots + \frac{\partial f}{\partial y_k}(\mathbf{b})\frac{\partial g_k}{\partial x_2}(\mathbf{a}), \\
& \ldots \qquad (10.3) \\
\frac{\partial h}{\partial x_n}(\mathbf{a}) &= \frac{\partial f}{\partial y_1}(\mathbf{b})\frac{\partial g_1}{\partial x_n}(\mathbf{a}) + \frac{\partial f}{\partial y_2}(\mathbf{b})\frac{\partial g_2}{\partial x_n}(\mathbf{a}) + \cdots + \frac{\partial f}{\partial y_k}(\mathbf{b})\frac{\partial g_k}{\partial x_n}(\mathbf{a}).
\end{aligned}
$$

If any point $\mathbf{a} \in \mathrm{D}_{\mathbf{g}} = \mathrm{D}_h$ can be put to the equalities (11.3), we can write the corresponding formulas for partial derivatives as functions:

$$\begin{aligned}
\frac{\partial h}{\partial x_1} &= \frac{\partial f}{\partial y_1}\frac{\partial g_1}{\partial x_1} + \frac{\partial f}{\partial y_2}\frac{\partial g_2}{\partial x_1} + \ldots + \frac{\partial f}{\partial y_k}\frac{\partial g_k}{\partial x_1}\,, \\
\frac{\partial h}{\partial x_2} &= \frac{\partial f}{\partial y_1}\frac{\partial g_1}{\partial x_2} + \frac{\partial f}{\partial y_2}\frac{\partial g_2}{\partial x_2} + \ldots + \frac{\partial f}{\partial y_k}\frac{\partial g_k}{\partial x_2}\,, \\
\ldots \\
\frac{\partial h}{\partial x_n} &= \frac{\partial f}{\partial y_1}\frac{\partial g_1}{\partial x_n} + \frac{\partial f}{\partial y_2}\frac{\partial g_2}{\partial x_n} + \ldots + \frac{\partial f}{\partial y_k}\frac{\partial g_k}{\partial x_n}\,.
\end{aligned} \tag{10.4}$$

☛ ***Example 2.*** Show that any function $F(x,y,z) = xf\left(\frac{y}{x},\frac{z}{x}\right)$, where $f(u,v)$ has continuous partial derivatives, satisfies the equation

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = F.$$

Solution. The exterior function $f(u,v)$ is a function of two variables,

$$u = g_1(x,y,z) = \frac{y}{x},\ \ v = g_2(x,y,z) = \frac{z}{x}.$$

To calculate partial derivatives of the resulting function $F(x,y)$, we will use the rule for the derivative of a product, for the derivative of the second factor we will use the formula (11.3).

$$\frac{\partial F}{\partial x} = f\left(\frac{y}{x},\frac{z}{x}\right) + x\left(\frac{\partial f}{\partial u}\frac{\partial g_1}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial g_2}{\partial x}\right) = f\left(\frac{y}{x},\frac{z}{x}\right) - \frac{y}{x}\frac{\partial f}{\partial u} - \frac{z}{x}\frac{\partial f}{\partial v},$$

$$\frac{\partial F}{\partial y} = x\left(\frac{\partial f}{\partial u}\frac{\partial g_1}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial g_2}{\partial y}\right) = \frac{\partial f}{\partial u},$$

$$\frac{\partial F}{\partial z} = x\left(\frac{\partial f}{\partial u}\frac{\partial g_1}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial g_2}{\partial z}\right) = \frac{\partial f}{\partial v}.$$

Now it is sufficient to multiply the equalities by variables x, y a z, respectively, and sum together.

☛ ***Example 3.*** Find the differential of a composite function

$$F(x,y) = f(u,v,w)$$

where

$$u = u(x,y) = x^2 + y^2\,,\ v = v(x,y) = x^2 - y^2\,,\ w = w(x,y) = 2xy.$$

Solution.

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}2x + \frac{\partial f}{\partial v}2x + \frac{\partial f}{\partial w}2y,$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}2y - \frac{\partial f}{\partial v}2y + \frac{\partial f}{\partial w}2x.$$

Thus

$$\mathrm{d}F((x,y),(h,k)) = 2\left(x\frac{\partial f}{\partial u} + x\frac{\partial f}{\partial v} + y\frac{\partial f}{\partial w}\right)h + 2\left(y\frac{\partial f}{\partial u} - y\frac{\partial f}{\partial v} + x\frac{\partial f}{\partial w}\right)k.$$

☛ ***Example 4.*** Transform differential operators

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}, \qquad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

from variables x, y to variables $u = x^2 - y^2,\ v = x - y$.

Solution. Denote

$$f(x,y) = h(u(x,y), v(x,y)) = h(x^2 - y^2, x - y)$$

and express partial derivatives of a composite function:

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial h}{\partial u}2x + \frac{\partial h}{\partial v},$$

$$\frac{\partial f}{\partial y} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial h}{\partial u}(-2y) + \frac{\partial h}{\partial v}(-1).$$

Thus

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y} = 2\frac{\partial h}{\partial u}(x-y) = 2v\frac{\partial h}{\partial u},$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = 2x^2\frac{\partial h}{\partial u}+x\frac{\partial h}{\partial v}-2y^2\frac{\partial h}{\partial u}-y\frac{\partial h}{\partial v} =$$

$$= 2(x^2-y^2)\frac{\partial h}{\partial u}+(x-y)\frac{\partial h}{\partial v} = 2u\frac{\partial h}{\partial u}+v\frac{\partial h}{\partial v}.$$

☛ ***Example 5.*** Transform the differential operator

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$$

from Cartesian coordinates x, y to polar coordinates r, φ, where $x = r\cos\varphi,\ y = r\sin\varphi$.

Solution. Denote

$$h(r,\varphi) = f(r\cos\varphi, r\sin\varphi).$$

Thus
$$\frac{\partial h}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\varphi + \frac{\partial f}{\partial y}\sin\varphi,$$
$$\frac{\partial h}{\partial \varphi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varphi} = \frac{\partial f}{\partial x}(-r\sin\varphi) + \frac{\partial f}{\partial y}r\cos\varphi.$$
Multiply the first equation by $\sin\varphi$, resp. $\cos\varphi$, the second one by $(\cos\varphi)/r$, resp. $(-\sin\varphi)/r$, and add together. We get

$$\frac{\partial h}{\partial r}\sin\varphi + \frac{1}{r}\frac{\partial h}{\partial \varphi}\cos\varphi = \frac{\partial f}{\partial y}.$$

$$\frac{\partial h}{\partial r}\cos\varphi - \frac{1}{r}\frac{\partial h}{\partial \varphi}\sin\varphi = \frac{\partial f}{\partial x}.$$

Now we can put the result to the differential operator. We obtain

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{\partial h}{\partial r}r(\cos^2\varphi + \sin^2\varphi) = r\frac{\partial h}{\partial r}.$$

Exercises

1. Let f be an arbitrary function that has a continuous derivative. Show that any function $F(x,y) = xf(x^2+y^2)$ satisfies the equation
$$y^2\frac{\partial F}{\partial x} - xy\frac{\partial F}{\partial y} = \frac{y^2}{x}F.$$

2. Let f be an arbitrary function that has a continuous derivative. Show that any function $F(x,y) = f(x^2-y^2)$ satisfies the equation
$$y\frac{\partial F}{\partial x} + x\frac{\partial F}{\partial y} = 0.$$

3. Let f be an arbitrary nonzero function that has a continuous derivative. Show that any function $F(x,y) = \dfrac{y}{f(x^2-y^2)}$ satisfies the equation
$$y^2\frac{\partial F}{\partial x} + xy\frac{\partial F}{\partial y} = xF.$$

4. Let $f(u, v)$ be an arbitrary function that has continuous partial derivatives. Show that for any natural number n, the function $F(x, y, z) = x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$ satisfies the equation

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = nF.$$

5. Let $f(u, v)$ be an arbitrary function that has continuous partial derivatives. Show that the function

$$F(x, y, z) = \frac{xy}{z}\ln x + xf\left(\frac{y}{x}, \frac{z}{x}\right)$$

satisfies the equation

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = F + \frac{xy}{z}.$$

6. Find the differential of a composite function $F(x, y) = f(t)$, where

   (a) $t = x + y\,;$ $\quad [\mathrm{d}F((x, y), (\mathrm{d}x, \mathrm{d}y)) = f'(t)\,\mathrm{d}x + f'(t)\,\mathrm{d}y.]$

(b) $t = \frac{y}{x}$. $\left[\mathrm{d}F((x,y),(\,\mathrm{d}x,\ \mathrm{d}y)) = -\frac{yf'(t)}{x^2}\,\mathrm{d}x + \frac{f'(t)}{x}\,\mathrm{d}y.\right]$

7. Find the differential of a composite function $F(x,y,z) = f(t)$, where

   (a) $t = x^2 + y^2 + z^2$; $[\mathrm{d}F((x,y,z),(\,\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z)) = 2f'(t)x\,\mathrm{d}x + 2f'(t)y\,\mathrm{d}y + 2f'(t)z\,\mathrm{d}z.]$

   (b) $t = xyz$. $[\mathrm{d}F((x,y,z),(\,\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z)) = f'(t)yz\,\mathrm{d}x + f'(t)xz\,\mathrm{d}y + f'(t)xy\,\mathrm{d}z.]$

8. Find the differential of a composite function $F(x,y) = f(u,v)$, where $f(u,v) = u + v$, $u = e^{xy}$, $v = e^{-xy}$.

   $[\mathrm{d}F((x,y),(\,\mathrm{d}x,\ \mathrm{d}y)) = (ye^{xy} - ye^{-xy})\,\mathrm{d}x + (xe^{xy} - xe^{-xy})\,\mathrm{d}y\,.]$

9. Find the differential of a composite function $F(x,y) = f(u,v,w)$, where $f(u,v,w) = u + v^2 + w^3$, and $u = u(x,y)$, $v = v(x,y)$, $w = w(x,y)$ have continuous partial derivatives.

   $\left[\mathrm{d}F((x,y),(\,\mathrm{d}x,\ \mathrm{d}y)) = \left(\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} + 3w^2\frac{\partial w}{\partial x}\right)\,\mathrm{d}x + \left(\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} + 3w^2\frac{\partial w}{\partial y}\right)\,\mathrm{d}y.\right]$