CHAPTER 11

HIGHER-ORDER DIFFERENTIALS

Higher-order partial derivatives

If we consider a partial derivative of a function f with respect to a variable x_i at a general point $(x_1, x_2, \ldots, x_n)$, it is again a function of variables $(x_1, x_2, \ldots, x_n)$ and we can consider its partial derivative with respect to j-th variable as a function

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{x}) = f''_{x_i x_j}(\boldsymbol{x}) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}(\boldsymbol{x})\right). \tag{11.1}$$

This function is called **partial derivative of second order of a function f with respect to variables x_i and x_j.**

Similarly we can proceed further. In general, we get **a k-th order partial derivative of f with respect to variables** $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$:

$$\frac{\partial^k f}{\partial x_{i_k} \partial x_{i_{k-1}} \ldots \partial x_{i_1}}(\boldsymbol{x}) \qquad \textbf{or} \qquad f^{(k)}_{x_{i_1} x_{i_2} \ldots x_{i_k}}(\boldsymbol{x}). \tag{11.2}$$

Theorem 1 (Interchangeability of partial derivatives).
If a function $f\colon \mathrm{D}_f \subset \mathbb{R}^n \to \mathbb{R}^1$ has a continuous partial derivative of the second order with respect to x_i and x_j in some neighbourhood of a point $\boldsymbol{a} \in \mathrm{D}_f^0$, then there exists the partial derivative of the second order of the function f with respect to x_j and x_i at $\boldsymbol{a}$ and both derivatives are equal:

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}). \qquad (11.3)$$

We also say that under the stated conditions, partial derivatives are **interchangable**.

More general:

Theorem 2 (Interchangeability of partial derivatives).
If a function $f\colon \mathrm{D}_f \subset \mathbb{R}^n \to \mathbb{R}^1$ has continuous partial derivatives up to the k-th order on an open set $M \subset \mathrm{D}_f$, then these partial derivatives do not depend on the order of variables but only on the number of differentiations with respect to individual variables.

Function sets $\mathbb{C}^k(M)$

The set $\mathbb{C}^k(M)$ is a set of functions $f\colon \mathrm{D}_f \subset \mathbb{R}^n \to \mathbb{R}^1$ that have continuous partial derivatives up to the k-th order on the given open subset $M \subset \mathrm{D}_f$. The set $\mathbb{C}^k(M)$ consists of functions that are continuous on M.

Higher-order partial derivatives of a composite function

Let us now return to partial derivatives of composite functions and ask how to find their higher-order derivatives.

Suppose that a function $f\colon \mathrm{D}_f \subset \mathbb{R}^2 \to \mathbb{R}^1$ and a vector function $\boldsymbol{g} = (g_1, g_2)\colon \mathrm{D}_{\boldsymbol{g}} \subset \mathbb{R}^2 \to \mathbb{R}^2$ are given such that

$$(x, y) = (g_1(u, v), g_2(u, v)) \in \mathrm{D}_f \text{ for all } (u, v) \in \mathrm{D}_{\boldsymbol{g}}.$$

Suppose that the functions g_1, g_2 have continuous partial derivatives of the second order at a point (u, v) and the function f has continuous partial derivatives of the second order at the point $(x, y) = (g_1(u, v), g_2(u, v))$. Then, as we know, the composite function $h = f \circ \boldsymbol{g}$ satisfies the equation

$$\frac{\partial h}{\partial u}(u,v) = \tag{11.4}$$

$$\frac{\partial f}{\partial x}(g_1(u,v),g_2(u,v))\frac{\partial g_1}{\partial u}(u,v)+\frac{\partial f}{\partial y}(g_1(u,v),g_2(u,v))\frac{\partial g_2}{\partial u}(u,v).$$

Using the formula for the derivative of a composite function, we can find the second derivatives:

$$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}(g_1(u,v),g_2(u,v))\frac{\partial g_1}{\partial u}(u,v)\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}(g_1(u,v),g_2(u,v))\right)\cdot\frac{\partial g_1}{\partial u}(u,v)+$$

$$+\frac{\partial f}{\partial x}(g_1(u,v),g_2(u,v))\cdot\frac{\partial}{\partial u}\left(\frac{\partial g_1}{\partial u}(u,v)\right) = \left[\frac{\partial^2 f}{\partial x^2}(g_1(u,v),g_2(u,v))\frac{\partial g_1}{\partial u}(u,v)+\right.$$

$$\left.+\frac{\partial^2 f}{\partial y\partial x}(g_1(u,v),g_2(u,v))\frac{\partial g_2}{\partial u}(u,v)\right]\frac{\partial g_1}{\partial u}(u,v)+\frac{\partial f}{\partial x}(g_1(u,v),g_2(u,v))\frac{\partial^2 g_1}{\partial u^2}(u,v).$$

Shortly,

$$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\frac{\partial g_1}{\partial u}\right) = \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial g_1}{\partial u}\right)^2+\frac{\partial^2 f}{\partial x\partial y}\frac{\partial g_1}{\partial u}\frac{\partial g_2}{\partial u}+\frac{\partial f}{\partial x}\frac{\partial^2 g_1}{\partial u^2}.$$

Analogously we get

$$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\frac{\partial g_2}{\partial u}\right)=\frac{\partial^2 f}{\partial x\partial y}\frac{\partial g_1}{\partial u}\frac{\partial g_2}{\partial u}+\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial g_2}{\partial u}\right)^2+\frac{\partial f}{\partial y}\frac{\partial^2 g_2}{\partial u^2},$$

thus

$$\frac{\partial^2 h}{\partial u^2}=\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\frac{\partial g_1}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial g_2}{\partial u}\right)=$$

$$=\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial g_1}{\partial u}\right)^2+\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial g_2}{\partial u}\right)^2+2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial g_1}{\partial u}\frac{\partial g_2}{\partial u}+\frac{\partial f}{\partial x}\frac{\partial^2 g_1}{\partial u^2}+\frac{\partial f}{\partial y}\frac{\partial^2 g_2}{\partial u^2}.$$

Similarly,

$$\frac{\partial^2 h}{\partial v^2}=\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial g_1}{\partial v}\right)^2+\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial g_2}{\partial v}\right)^2+2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial g_1}{\partial v}\frac{\partial g_2}{\partial v}+\frac{\partial f}{\partial x}\frac{\partial^2 g_1}{\partial v^2}+\frac{\partial f}{\partial y}\frac{\partial^2 g_2}{\partial v^2},$$

$$\frac{\partial^2 h}{\partial u\partial v}=\frac{\partial^2 f}{\partial x^2}\frac{\partial g_1}{\partial u}\frac{\partial g_1}{\partial v}+\frac{\partial^2 f}{\partial y^2}\frac{\partial g_2}{\partial u}\frac{\partial g_2}{\partial v}+$$

$$+\frac{\partial^2 f}{\partial x\partial y}\left(\frac{\partial g_1}{\partial u}\frac{\partial g_2}{\partial v}+\frac{\partial g_1}{\partial v}\frac{\partial g_2}{\partial u}\right)+\frac{\partial f}{\partial x}\frac{\partial^2 g_1}{\partial u\partial v}+\frac{\partial f}{\partial y}\frac{\partial^2 g_2}{\partial u\partial v}.$$

☛ ***Example 1.*** Find all partial derivatives of the function

$$f(x,y) = x^5 + 3x^3y^2 - 5xy^4$$

up to the third order.

Solution. For all $(x,y) \in \mathbb{R}^2$:

$$f'_x(x,y) = 5x^4 + 9x^2y^2 - 5y^4\,, \quad f'_y(x,y) = 6x^3y - 20xy^3;$$

$$f''_{xx}(x,y) = 20x^3 + 18xy^2\,, f''_{yy}(x,y) = 6x^3 - 60xy^2\,,$$

$$f''_{xy} = f''_{yx}(x,y) = 18x^2y - 20y^3;$$

$$f'''_{xxx}(x,y) = 60x^2{+}18y^2\,, f'''_{xxy}(x,y) = f'''_{xyx}(x,y) = f'''_{yxx}(x,y) = 36xy\,,$$

$$f'''_{yyy}(x,y) = -120xy\,, f'''_{yyx}(x,y) = f'''_{yxy}(x,y) = f'''_{xyy}(x,y) = 18x^2{-}60y^2.$$

☛ ***Example 2.*** Find the partial derivative $f^{(4)}_{xyzz}(x,y,z)$ of the function $f(x,y,z) = 3x^5y^2z^3 - x\,e^{yz}$.

Solution. For all $(x,y) \in \mathbb{R}^2$:

$$\begin{aligned}
f'_x(x,y,z) &= 15x^4y^2z^3 - e^{yz}\,, \\
f''_{xy}(x,y,z) &= 30x^4yz^3 - z\,e^{yz}\,, \\
f'''_{xyz}(x,y,z) &= 90x^4yz^2 - e^{yz} - yz\,e^{yz}\,, \\
f^{(4)}_{xyzz}(x,y,z) &= 180x^4yz - 2y\,e^{yz} - y^2z\,e^{yz}\,.
\end{aligned}$$

Exercises

1. Let $f(x,y) = \ln(x+y),\ (x,y) \in \mathrm{D}_f = \{(x,y) \in \mathbb{R}^2 \mid y > -x\}$. Show that

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = -\frac{1}{(x+y)^2}\,,$$

and for any $k \in \mathbb{N}$ and any $r = 0, 1, \ldots, k$:

$$\frac{\partial^k f}{\partial x^{k-r} \partial y^r}(x,y) = (-1)^{k-1}(k-1)!\,\frac{1}{(x+y)^k}\,.$$

2. Let $f(x,y) = \mathrm{e}^{xy},\ (x,y) \in \mathbb{R}^2$. Show that

$$\frac{\partial f}{\partial x}(x,y) = y\mathrm{e}^{xy}, \quad \frac{\partial f}{\partial y}(x,y) = x\mathrm{e}^{xy}, \quad \frac{\partial^2 f}{\partial x^2}(x,y) = y^2\mathrm{e}^{xy},$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = x^2\mathrm{e}^{xy}, \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = (1+xy)\mathrm{e}^{xy}.$$

Show that for any $k \in \mathbb{N}$, the following equation holds:

$$\frac{\partial^k f}{\partial x^k}(x,y) = y^k\mathrm{e}^{xy}, \qquad \frac{\partial^k f}{\partial y^k}(x,y) = x^k\mathrm{e}^{xy}.$$

3. Let $f(x,y) = e^{x+y},\ (x,y) \in \mathbb{R}^2$. Show that

$$\frac{\partial^k f}{\partial x^k}(x,y) = \frac{\partial^k f}{\partial y^k}(x,y) = \frac{\partial^k f}{\partial x^{k-r} \partial y^r}(x,y) = e^{x+y},\ r = 0,1,\ldots,k; \quad k \in$$

4. Let $f(x,y) = \mathrm{e}^{x+y^2},\ (x,y) \in \mathbb{R}^2$. Show that for any $k \in \mathbb{N}$:

$$\frac{\partial f}{\partial x}(x,y) = \mathrm{e}^{x+y^2}, \quad \frac{\partial f}{\partial y}(x,y) = 2y\mathrm{e}^{x+y^2},$$

$$\frac{\partial^k f}{\partial x^k}(x,y) = \mathrm{e}^{x+y^2}, \quad \frac{\partial^k f}{\partial x^{k-1} \partial y}(x,y) = 2y\mathrm{e}^{x+y^2} = \frac{\partial f}{\partial y}(x,y).$$

5. Show that the function f satisfies the given equation.

(a) $f(x,y) = \ln(x^2+y^2)\,,\qquad f''_{xx} + f''_{yy} = 0\,.$

(b) $f(x,y) = \dfrac{y}{y^2 - a^2x^2}\,,\qquad f''_{xx} - a^2 f''_{yy} = 0\,.$

(c) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}\,,\qquad f''_{xx} + f''_{yy} + f''_{zz} = 0\,.$

(d) $f(x,y,z) = \ln\sqrt{x^2+y^2+z^2}\,,\qquad f''_{xx}+f''_{yy}+f''_{zz} = \dfrac{1}{x^2+y^2+z^2}\,.$

Differential of the k-th order

In the differential calculus of functions of one variable, we have introduced differentials of higher orders to improve an approximation of functions. Similarly we can proceed also in the case of functions of more variables.

Definition 1. Suppose that f has partial derivatives of the second order in a neighbourhood of a point $\boldsymbol{a}$ and these derivatives are continuous at $\boldsymbol{a}$. Then **the differential of the second order of the function f at the point $\boldsymbol{a}$** is defined by the formula

$$\mathrm{d}^2 f(\boldsymbol{a},\boldsymbol{h}) = \frac{\partial^2 f}{\partial x_1^2}(\boldsymbol{a})h_1^2 + \frac{\partial^2 f}{\partial x_2^2}(\boldsymbol{a})h_2^2 + \ldots + \frac{\partial^2 f}{\partial x_n^2}(\boldsymbol{a})h_n^2 + \tag{11.5}$$

$$+ \quad 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(\boldsymbol{a})h_1 h_2 + \ldots + 2\frac{\partial^2 f}{\partial x_{n-1}\partial x_n}(\boldsymbol{a})h_{n-1}h_n,$$

where $\boldsymbol{h} = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$.

Specially, for functions of two variables, we have

$$\mathrm{d}^2 f(\boldsymbol{a},\boldsymbol{h}) = \frac{\partial^2 f}{\partial x_1^2}(\boldsymbol{a})h_1^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(\boldsymbol{a})h_1 h_2 + \frac{\partial^2 f}{\partial x_2^2}(\boldsymbol{a})h_2^2. \tag{11.6}$$

This differential can also be written in the form

$$\mathrm{d}^2 f(\boldsymbol{a}, \boldsymbol{x}-\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_1^2}(\boldsymbol{a})(x_1-a_1)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(\boldsymbol{a})(x_1-a_1)(x_2-a_2) + \\ + \frac{\partial^2 f}{\partial x_2^2}(\boldsymbol{a})(x_2-a_2)^2, \quad \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2, \tag{11.7}$$

or in the form

$$\mathrm{d}^2 f(\boldsymbol{a},\, \mathbf{d}\boldsymbol{x}) = \frac{\partial^2 f}{\partial x_1^2}(\boldsymbol{a})\,\mathrm{d}x_1^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(\boldsymbol{a})\,\mathrm{d}x_1\,\mathrm{d}x_2 + \frac{\partial^2 f}{\partial x_2^2}(\boldsymbol{a})\,\mathrm{d}x_2^2, \\ \mathbf{d}\boldsymbol{x} = (\,\mathrm{d}x_1,\,\mathrm{d}x_2) \in \mathbb{R}^2. \tag{11.8}$$

Analogously we can also introduce differentials of higher orders. For functions of two variables, we can write

$$\mathrm{d}^k f(\boldsymbol{a}, \boldsymbol{h}) = \left(\frac{\partial}{\partial x}h_1 + \frac{\partial}{\partial y}h_2\right)^k f(\boldsymbol{a}) = \sum_{j=0}^{k} \binom{k}{j} \frac{\partial^k f(\boldsymbol{a})}{\partial x^{k-j} \partial y^j} h_1^{k-j} h_2^j. \tag{11.9}$$

Obviously,

$$\mathrm{d}^1 f(\boldsymbol{a},\boldsymbol{h}) = \frac{\partial f}{\partial x}(\boldsymbol{a})h_1 + \frac{\partial f}{\partial y}(\boldsymbol{a})h_2 = \mathrm{d}f(\boldsymbol{a},\boldsymbol{h}),$$

$$\mathrm{d}^2 f(\boldsymbol{a},\boldsymbol{h}) = \frac{\partial^2 f}{\partial x^2}(\boldsymbol{a})h_1^2 + 2\frac{\partial^2 f}{\partial x \partial y}(\boldsymbol{a})h_1 h_2 + \frac{\partial^2 f}{\partial y^2}(\boldsymbol{a})h_2^2,$$

$$\mathrm{d}^3 f(\boldsymbol{a},\boldsymbol{h}) = \frac{\partial^3 f}{\partial x^3}(\boldsymbol{a})h_1^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}(\boldsymbol{a})h_1^2 h_2 + 3\frac{\partial^3 f}{\partial x \partial y^2}(\boldsymbol{a})h_1 h_2^2 + \frac{\partial^3 f}{\partial y^3}(\boldsymbol{a})h_2^3,$$

etc.

For more variables:

$$\mathrm{d}^k f(\boldsymbol{a},\boldsymbol{h}) = \sum_{\substack{r_1, r_2, \cdots, r_n \\ r_1 + r_2 + \cdots + r_n = k}} \frac{k!}{r_1! r_2! \cdots r_n!} \, \frac{\partial^k f(\boldsymbol{a})}{\partial x_1^{r_1} \partial x_2^{r_2} \cdots \partial x_n^{r_n}} h_1^{r_1} h_2^{r_2} \cdots h_n^{r_n} \tag{11.10}$$

For example:

$$\begin{aligned}
\mathrm{d}f(\boldsymbol{a},\boldsymbol{h}) &= \frac{\partial f}{\partial x}(\boldsymbol{a})h_1 + \frac{\partial f}{\partial y}(\boldsymbol{a})h_2 + \frac{\partial f}{\partial z}(\boldsymbol{a})h_3, \quad (h_1,h_2,h_3)\in\mathbb{R}^3.\\
\mathrm{d}^2 f(\boldsymbol{a},\boldsymbol{h}) &= \frac{\partial^2 f}{\partial x^2}(\boldsymbol{a})h_1^2 + \frac{\partial^2 f}{\partial y^2}(\boldsymbol{a})h_2^2 + \frac{\partial^2 f}{\partial z^2}(\boldsymbol{a})h_3^2 + 2\frac{\partial^2 f}{\partial x\partial y}(\boldsymbol{a})h_1h_2 +\\
&+ 2\frac{\partial^2 f}{\partial x\partial z}(\boldsymbol{a})h_1h_3 + 2\frac{\partial^2 f}{\partial y\partial z}(\boldsymbol{a})h_2h_3, \quad (h_1,h_2,h_3)\in\mathbb{R}^3.\\
\mathrm{d}^3 f(\boldsymbol{a},\boldsymbol{h}) &= \frac{\partial^3 f}{\partial x^3}(\boldsymbol{a})h_1^3 + \frac{\partial^3 f}{\partial y^3}(\boldsymbol{a})h_2^3 + \frac{\partial^3 f}{\partial z^3}(\boldsymbol{a})h_3^3 + 3\frac{\partial^3 f}{\partial x^2\partial y}(\boldsymbol{a})h_1^2h_2 +\\
&+ 3\frac{\partial^3 f}{\partial x^2\partial z}(\boldsymbol{a})h_1^2h_3 + 3\frac{\partial^3 f}{\partial x\partial y^2}(\boldsymbol{a})h_1h_2^2 + 3\frac{\partial^3 f}{\partial x\partial z^2}(\boldsymbol{a})h_1h_3^2 +\\
&+ 3\frac{\partial^3 f}{\partial y^2\partial z}(\boldsymbol{a})h_2^2h_3 + 3\frac{\partial^3 f}{\partial y\partial z^2}(\boldsymbol{a})h_2h_3^2 + 6\frac{\partial^3 f}{\partial x\partial y\partial z}(\boldsymbol{a})h_1h_2h_3\,.
\end{aligned}$$

Taylor polynomial

Definition 2. Taylor polynomial of degree k for a function f at a point a is defined by the formula

$$\mathsf{T}_f^k(\boldsymbol{a},\boldsymbol{h}) = f(\boldsymbol{a}) + \mathrm{d}f(\boldsymbol{a},\boldsymbol{h}) + \frac{1}{2!}\mathrm{d}^2 f(\boldsymbol{a},\boldsymbol{h}) + \cdots + \frac{1}{k!}\mathrm{d}^k f(\boldsymbol{a},\boldsymbol{h}),$$

$$\boldsymbol{h} \in \mathbb{R}^n\,, \tag{11.11}$$

or (in a different notation)

$$\mathsf{T}_f^k(\boldsymbol{a},\boldsymbol{x}-\boldsymbol{a}) = f(\boldsymbol{a}) + \mathrm{d}f(\boldsymbol{a},\boldsymbol{x}-\boldsymbol{a}) + \frac{1}{2!}\mathrm{d}^2 f(\boldsymbol{a},\boldsymbol{x}-\boldsymbol{a}) + \cdots +$$

$$+\frac{1}{k!}\mathrm{d}^k f(\boldsymbol{a},\boldsymbol{x}-\boldsymbol{a}),\ \boldsymbol{x} \in \mathbb{R}^n\,. \tag{11.12}$$

Taylor polynomial $\mathsf{T}_f^k(\boldsymbol{a},\boldsymbol{x}-\boldsymbol{a})$ is again used for an approximation of a function f in the neighbourhood of a point $\boldsymbol{a}$. This approximation is based on the following theorem.

Theorem 3 (Taylor's theorem). *Let* $f\colon \mathsf{D}_f \subset \mathbb{R}^n \to \mathbb{R}^1$, $f \in \mathbb{C}^{k+1}$ *on an open set* $M \subset \mathsf{D}_f$. *Let* $\boldsymbol{a}$ *and* $\boldsymbol{a}+\boldsymbol{h}$ *be two points of* M *such that the whole line segment with the end points* $\boldsymbol{a}$ *and* $\boldsymbol{a}+\boldsymbol{h}$ *lies in* M. *Then there exists a point* $\boldsymbol{\xi}$ *on this line segment such that*

$$f(\boldsymbol{a}+\boldsymbol{h}) = f(\boldsymbol{a}) + \mathrm{d}f(\boldsymbol{a},\boldsymbol{h}) + \frac{1}{2!}\mathrm{d}^2 f(\boldsymbol{a},\boldsymbol{h}) + \ldots + \frac{1}{k!}\mathrm{d}^k f(\boldsymbol{a},\boldsymbol{h}) + R^{k+1}(\boldsymbol{h})\,, \tag{11.13}$$

where

$$R^{k+1}(\boldsymbol{h}) = \frac{1}{(k+1)!}\mathrm{d}^{k+1} f(\boldsymbol{\xi},\boldsymbol{h})\,. \tag{11.14}$$

The equality (11.13) is called **Taylor's formula** *and the number (11.14) is called* **a remainder in the Lagrange form**.

☛ ***Example 3.*** Find the differential of the first, second and third order for the function $f(x,y) = e^x \cos y$ at $\boldsymbol{a} = (0,0)$.
Solution.

$$
\begin{array}{rclrcr}
f'_x(x,y) & = & e^x \cos y\,, & f'_x(0,0) & = & 1\,,\\
f'_y(x,y) & = & -e^x \sin y\,, & f'_y(0,0) & = & 0\,,\\
f''_{xx}(x,y) & = & e^x \cos y\,, & f''_{xx}(0,0) & = & 1\,,\\
f''_{xy}(x,y) & = & -e^x \sin y\,, & f''_{xy}(0,0) & = & 0\,,\\
f''_{yy}(x,y) & = & -e^x \cos y\,, & f''_{yy}(0,0) & = & -1\,,\\
f'''_{xxx}(x,y) & = & e^x \cos y\,, & f'''_{xxx}(0,0) & = & 1\,,\\
f'''_{xxy}(x,y) & = & -e^x \sin y\,, & f'''_{xxy}(0,0) & = & 0\,,\\
f'''_{xyy}(x,y) & = & -e^x \cos y\,, & f'''_{xyy}(0,0) & = & -1\,,\\
f'''_{yyy}(x,y) & = & e^x \sin y\,, & f'''_{yyy}(0,0) & = & 0\,.
\end{array}
$$

The corresponding differentials are now equal to

$$
\begin{array}{rcll}
\mathrm{d}^1 f((0,0),(h_1,h_2)) & = & h_1\,, & (h_1,h_2) \in \mathbb{R}^2\,;\\
\mathrm{d}^2 f((0,0),(h_1,h_2)) & = & h_1^2 - h_2^2\,, & (h_1,h_2) \in \mathbb{R}^2\,;\\
\mathrm{d}^3 f((0,0),(h_1,h_2)) & = & h_1^3 - 3h_1 h_2^2\,, & (h_1,h_2) \in \mathbb{R}^2\,.
\end{array}
$$

☛ ***Example 4.*** Using Taylor polynomial of degree 2, find the approximate value of the function $f(x,y) = x^y$ at $(1{,}05; 3{,}02)$.

Solution. Let us approximate the function $f(x,y) = x^y$ in the neighbourhood of $\boldsymbol{a} = (1,3)$ using a quadratic function $\mathrm{d}^2 f(\boldsymbol{a},\boldsymbol{h})$. Partial derivatives are equal to

$$
\begin{aligned}
f'_x(\boldsymbol{a}) &= y x^{y-1}\Big|_{\boldsymbol{a}} &&= 3\,,\\
f'_y(\boldsymbol{a}) &= x^y \ln x\Big|_{\boldsymbol{a}} &&= 0\,,\\
f''_{xx}(\boldsymbol{a}) &= y(y-1)x^{y-2}\Big|_{\boldsymbol{a}} &&= 6\,,\\
f''_{xy}(\boldsymbol{a}) &= x^{y-1} + y x^{y-1}\ln x\Big|_{\boldsymbol{a}} &&= 1\,,\\
f''_{yy}(\boldsymbol{a}) &= x^y \ln^2 x\Big|_{\boldsymbol{a}} &&= 0\,.
\end{aligned}
$$

Using Taylor formula, we obtain (the exact value is $1{,}1588$)

$$
f(1{,}05;3{,}02) \doteq 1+3\cdot 0,05+0\cdot 0,02+\frac{1}{2}(6\cdot 0{,}05^2+2\cdot 1\cdot 0{,}05\cdot 0{,}02+0\cdot 0{,}02^2) =
$$

$$
= 1{,}1585.
$$

☛ ***Example 5.*** Find the Taylor polynomial $\mathrm{T}_f^3(\boldsymbol{a}, \boldsymbol{h})$ of the function $f(x, y) = x \sin^2 y$ at the point $\boldsymbol{a} = (1, \pi/2)$.

Solution. We are looking for the Taylor polynomial of the third degree. We need the differentials $\mathrm{d}^k f(\boldsymbol{a}, \boldsymbol{h})$ for $k = 1, 2, 3$.

$$\frac{\partial f}{\partial x}(x, y) = \sin^2 y, \quad \frac{\partial f}{\partial y}(x, y) = 2x \sin y \cos y = x \sin 2y,$$

$$\frac{\partial^2 f}{\partial x^2}(x, y) = 0, \quad \frac{\partial^2 f}{\partial x \partial y}(x, y) = \sin 2y, \quad \frac{\partial^2 f}{\partial y^2}(x, y) = 2x \cos 2y,$$

$$\frac{\partial^3 f}{\partial x^3}(x, y) = 0, \ \frac{\partial^3 f}{\partial x^2 \partial y}(x, y) = 0,$$

$$\frac{\partial^3 f}{\partial x \partial y^2}(x, y) = 2 \cos 2y, \ \frac{\partial^3 f}{\partial y^3}(x, y) = -4x \sin 2y.$$

For $x = 1, \ y = \pi/2$ we get

$$\mathrm{d}f(\boldsymbol{a}, \boldsymbol{h}) = \left(\sin^2 \frac{\pi}{2}\right) h_1 + (\sin \pi) h_2 = h_1,$$

$$\mathrm{d}^2 f(\boldsymbol{a}, \boldsymbol{h}) = 0 \cdot h_1^2 + (2 \sin \pi) h_1 h_2 + (2 \cos \pi) h_2^2 = -2h_2^2,$$

$$d^3 f(\boldsymbol{a}, \boldsymbol{h}) = 0 \cdot h_1^3 + 3 \cdot 0 \cdot h_1^2 h_2 + 3(2\cos\pi) h_1 h_2^2 + (-4\sin\pi) h_2^3 = -6 h_1 h_2^2,$$

thus

$$\mathsf{T}_f^3(\boldsymbol{a}, \boldsymbol{h}) = f(\boldsymbol{a}) + \mathrm{d}f(\boldsymbol{a}, \boldsymbol{h}) + \frac{1}{2!}\mathrm{d}^2 f(\boldsymbol{a}, \boldsymbol{h}) + \frac{1}{3!}\mathrm{d}^3 f(\boldsymbol{a}, \boldsymbol{h}) =$$

$$= 1 + h_1 - h_2^2 - h_1 h_2^2, \qquad (h_1, h_2) \in \mathbb{R}^2.$$

Using the notation from (11.7), we can also write

$$\mathrm{d}f(\boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a}) = x_1 - 1,$$

$$\mathrm{d}^2 f(\boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a}) = -2(x_2 - \pi/2)^2,$$

$$\mathrm{d}^3 f(\boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a}) = -6(x_1 - 1)(x_2 - \pi/2)^2,$$

so that

$$\mathsf{T}_f^3(\boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a}) = 1 + (x_1 - 1) - (x_2 - \pi/2)^2 - (x_1 - a_1)(x_2 - \pi/2)^2,$$

$$(x_1, x_2) \in \mathbb{R}^2.$$

☛ ***Example 6.*** Find the Taylor polynomial $\mathsf{T}_f^3(\boldsymbol{a},\boldsymbol{h})$ of the function $f(x,y) = x^2yz$ at the point $\boldsymbol{a} = (2,3,1)$.

Solution. $f(2,3,1) = 12$,

$$\mathrm{d}f(\boldsymbol{a},\boldsymbol{h}) = \frac{\partial f}{\partial x}(\boldsymbol{a})h_1 + \frac{\partial f}{\partial y}(\boldsymbol{a})h_2 + \frac{\partial f}{\partial z}(\boldsymbol{a})h_3 = 12h_1 + 4h_2 + 12h_3,$$

$$\mathrm{d}^2f(\boldsymbol{a},\boldsymbol{h}) = \frac{\partial^2 f}{\partial x^2}(\boldsymbol{a})h_1^2 + \frac{\partial^2 f}{\partial y^2}(\boldsymbol{a})h_2^2 + \frac{\partial^2 f}{\partial z^2}(\boldsymbol{a})h_3^2 + 2\frac{\partial^2 f}{\partial x\partial y}(\boldsymbol{a})h_1h_2 +$$

$$+2\frac{\partial^2 f}{\partial x\partial z}(\boldsymbol{a})h_1h_3 + 2\frac{\partial^2 f}{\partial y\partial z}(\boldsymbol{a})h_2h_3 = 6h_1^2 + 8h_1h_2 + 12h_1h_3,$$

$$\mathrm{d}^3f(\boldsymbol{a},\boldsymbol{h}) = \frac{\partial^3 f}{\partial x^3}(\boldsymbol{a})h_1^3 + \frac{\partial^3 f}{\partial y^3}(\boldsymbol{a})h_2^3 + \frac{\partial^3 f}{\partial z^3}(\boldsymbol{a})h_3^3 +$$

$$+3\frac{\partial^3 f}{\partial x^2\partial y}(\boldsymbol{a})h_1^2h_2 + 3\frac{\partial^3 f}{\partial x^2\partial z}(\boldsymbol{a})h_1^2h_3 + 3\frac{\partial^3 f}{\partial x\partial y^2}(\boldsymbol{a})h_1h_2^2 + 3\frac{\partial^3 f}{\partial x\partial z^2}(\boldsymbol{a})h_1h_3^2 +$$

$$+3\frac{\partial^3 f}{\partial y^2\partial z}(\boldsymbol{a})h_2^2h_3 + 3\frac{\partial^3 f}{\partial y\partial z^2}(\boldsymbol{a})h_2h_3^2 + 6\frac{\partial^3 f}{\partial x\partial y\partial z}(\boldsymbol{a})h_1h_2h_3 =$$

$$= 6h_1^2h_2 + 18h_1^2h_3 + 24h_1h_2h_3\,, \quad \text{thus}$$

$$\begin{aligned}\mathsf{T}_f^3(\boldsymbol{a},\boldsymbol{h}) \;&=\; 12 + 12h_1 + 4h_2 + 12h_3 + 3h_1^2 + 4h_1h_2 + 6h_1h_3 + \\ &\quad +h_1^2h_2 + 3h_1^2h_3 + 4h_1h_2h_3, \quad (h_1,h_2,h_3) \in \mathbb{R}^3.\end{aligned}$$

Exercises

1. Find the k-th differential of the function f at the point $\boldsymbol{a}$.

   (a) $f(x,y) = \cos x \cos y\,,\ \boldsymbol{a} = (0,0)\,,\ k = 2$.

   $[\mathrm{d}^2 f(\boldsymbol{a},\boldsymbol{h}) = -h_1^2 - h_2^2]$

   (b) $f(x,y,z) = x^3 y^2 z\,,\ \boldsymbol{a} = (1,2,3)\,,\ k = 2$.

   $[\mathrm{d}^2 f(\boldsymbol{a},\boldsymbol{h}) = 72h_1^2 + 6h_2^2 + 72h_1h_2 + 24h_1h_3 + 8h_2h_3]$

   (c) $f(x,y) = y \ln x\,,\ \boldsymbol{a} = (1,3)\,,\ k = 3$.

   $[\mathrm{d}^3 f(\boldsymbol{a},\boldsymbol{h}) = 6h_1^3 - 3h_1^2h_2]$

2. Find the second differential of the given function f at an arbitrary point $\boldsymbol{x}$.

   (a) $f(x,y) = \sin(2x+y)$.

   $[\mathrm{d}^2 f(\boldsymbol{x},\boldsymbol{h}) = -\sin(2x+y)(4h_1^2 + 4h_1h_2 + h_2^2]$

   (b) $f(x,y) = \ln(x-y)$.

   $[\mathrm{d}^2 f(\boldsymbol{x},\boldsymbol{h}) = -\frac{1}{(x-y)^2}(h_1^2 + h_2^2 - 2h_1h_2.]$

(c) $f(x,y,z) = \sin(x+y+z)$. $[\mathrm{d}^2 f(\boldsymbol{x},\boldsymbol{h}) = -\sin(x+y+z)(h_1^2 + 2h_1h_2 + 2h_1h_3 + h_2^2 + 2h_2h_3 + h_3^2]$

(d) $f(x,y) = x\sin^2 y$.

$[\mathrm{d}^2 f(\boldsymbol{x},\ \mathbf{d}\boldsymbol{x}) = 2\sin 2y\,\mathrm{d}x\,\mathrm{d}y + 2x\cos 2y\,\mathrm{d}y^2]$

(e) $f(x,y) = xy^2 - x^2y$.

$[\mathrm{d}^2 f(\boldsymbol{x},\ \mathbf{d}\boldsymbol{x}) = -2y\,\mathrm{d}x^2 + 4(y-x)\,\mathrm{d}x\,\mathrm{d}y + 2x\,\mathrm{d}y^2]$

(f) $f(x,y,z) = xyz$.

$[\mathrm{d}^2 f(\boldsymbol{x},\ \mathbf{d}\boldsymbol{x}) = 2z\,\mathrm{d}x\,\mathrm{d}y + 2y\,\mathrm{d}x\,\mathrm{d}z + 2x\,\mathrm{d}y\,\mathrm{d}z.]$

3. Find the Taylor polynomial of the k-th degree for the given function f at the given point $\boldsymbol{a}$.

(a) $f(x,y) = \ln(1+x)\ln(1+y)$, $\boldsymbol{a} = (0,0)$, $k = 3$.

$[\mathsf{T}_f^3(\boldsymbol{a},\boldsymbol{h}) = h_1h_2 - \frac{1}{2}h_1^2h_2 - \frac{1}{2}h_1h_2^2]$

(b) $f(x,y,z) = \sin x \sin y \sin z$, $\boldsymbol{a} = (\pi/4, \pi/4, \pi/4)$, $k = 2$.

$[\mathsf{T}_f^2(\boldsymbol{a},\boldsymbol{h}) = \sqrt{2}/4(1+h_1+h_2+h_3-\frac{1}{2}h_1^2-\frac{1}{2}h_2^2-\frac{1}{2}h_3^2+h_1h_2+h_1h_3+h_2h_3]$

4. Approximate the function $f(x,y) = e^x \sin y$ in the neighbourhood of $\boldsymbol{o} = (0,0)$ by a Taylor polynomial of the third degree.

$$[e^x \sin y \doteq y + xy + \tfrac{1}{2}x^2y - \tfrac{1}{6}y^3]$$

5. Using Taylor polynomial of the second degree, find the approximate values of the following numbers.

   (a) $0{,}96^{2{,}02}$ $[0{,}920\ 8]$

   (b) $\sqrt[3]{0{,}98}\,\sqrt[4]{1{,}03}$ $[1{,}000\ 654\ 7]$