CHAPTER 6

HIGHER-ORDER DERIVATIVES

GRAPHS OF FUNCTIONS

Derivatives and higher-order differentials

Definition 1. If the derivative f' exists at every point of an interval I, then its derivative

$$f''(x) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) = (f')'(x)$$

is called **second derivative of f on I.**

For $n \in \mathbb{N}\,, n = 2, 3, \cdots$, **the n-th derivative** or **the derivative of the n-th order of the function f on I** is defined by a recursive formula

$$f^{(n)}(x) = \frac{\mathrm{d}^n f}{\mathrm{d}x^n}\,(x) = \frac{\mathrm{d}}{\mathrm{d}x}\,f^{(n-1)}(x)\,, \quad x \in I\,,$$

provided these derivatives exist on I.

If $I = D_f$ then $f^{(n)}$ is called **the n-th derivative of f.**

We will also denote $f^{(0)} \equiv f$.

☛ ***Example 1.***

Find the third derivative of the function

$$f(x) = 2x^5 - 3x^4 + 5x^3 + 3x^2 + 5x - 1\,.$$

Solution.

$$\begin{aligned} f'(x) &= 10x^4 - 12x^3 + 15x^2 + 6x + 5 \\ f''(x) &= 40x^3 - 36x^2 + 30x + 6 \\ f'''(x) &= 120x^3 - 72x + 30 \end{aligned}$$

Definition 2. If $f^{(n)}(x_0)$ exists, then

$$\mathrm{d}^n f(a;h) = f^{(n)}(a)h^n$$

is called **the differential of the n-th order of the function f in a point x_0.**

Theorem 1 (Leibniz rule)

Let functions f, g be such that their derivatives of orders $1, 2, \ldots, n$ exist at x_0. Then for every $n \in \mathbb{N}$ the following equation holds:

$$\left(f \cdot g\right)^{(n)}(x_0) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x_0) \cdot g^{(n-k)}(x_0)\,,$$

where $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$..

Remark. For example:

$$\begin{aligned} (uv)' &= uv' + u'v \\ (uv)'' &= uv'' + 2u'v' + u''v \\ (uv)''' &= uv''' + 3u'v'' + 3u''v' + u'''v \end{aligned}$$

Proof. By mathematical induction: for $n = 1$, we simply have a derivative of a product.
Suppose that the relation holds for n. By differentiating we get

$$\big(f(x)g(x)\big)^{(n+1)} = \Big(\big(f(x)g(x)\big)^{(n)}\Big)' =$$

$$= \left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)\right)' =$$

$$= \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)}(x) g^{(n-k)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k+1)}(x) =$$

$$= \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)}(x) g^{(n-k+1)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k+1)}(x) =$$

$$= f^{(n+1)}(x)g(x) + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k}\right] f^{(k)}(x) g^{(n-k+1)}(x) +$$

$$+ f(x) g^{(n+1)}(x) =$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(x) g^{(n+1-k)}(x)\,.$$

Remark: The set of all functions $f : X \to \mathbb{R}$ that have continuous n-th derivatives (and thus also all derivatives of a lower order) on a set X is denoted by $C_n(X)$.

The set of functions that are continuous on X is denoted by $C_0(X)$.

The set of all functions $f : X \to \mathbb{R}$ that have continuous derivatives of all orders on a set X is denoted by $C_\infty(X)$.

Obviously,

$$C_\infty(X) \subset \cdots \subset C_n(X) \subset C_{n-1}(X) \subset \cdots \subset C_1(X) \subset C_0(X)$$

for all $n \in \mathbb{N}$.

Taylor polynomial

Definition 3. Let $f'(x_0), \dots f^{(n)}(x_0)$ exist. The polynomial

$$\mathrm{T}^n f(x_0; h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}h^n$$

is called **Taylor polynomial of the n-th order of the function f in a point** x_0.

The following theorem enables an approximation of functions.

Theorem 2 (Taylor).

Let $f(x)$ be defined on $[a,b]$ and let derivatives of all orders be continuous on (a,b). Then for any two points $x, x_0 \in [a,b]$ there exists a point ξ between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \cdots$$

$$\cdots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n + R_{n+1}(x), \quad (6.1)$$

where

$$R_{n+1}(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1}. \quad (6.2)$$

The number $R_{n+1}(x)$ is called **a Lagrange remainder.**

☛ ***Example 2.***

Find the Taylor polynomial of $f(x) = \mathrm{e}^x$ at $x = 0$.

Solution.

For every $k \in \mathbb{N}$, $x \in \mathbb{R}$: $f^{(k)}(x) = (\mathrm{e}^x)^{(k)} = \mathrm{e}^x,\ f^{(k)}(0) = 1\,.$

Thus

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_{n+1}(x), \qquad R_{n+1}(x) = \frac{e^\xi}{(n+1)!}x^{n+1}\,,$$

where ξ lies between 0 and x.

Similarly:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + $$

$$+ (-1)^n \frac{\cos\xi}{(2n+1)!}x^{2n+1}\,, \qquad x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} +$$

$$+ (-1)^{n+1} \frac{\cos \xi}{(2n+2)!} x^{2n+2}\,, \quad x \in \mathbb{R}$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} +$$

$$(-1)^n \frac{1}{(n+1)(\xi+1)^{n+1}} x^{n+1}\,, \quad x > -1$$

Behaviour of a function – monotonicity

Let f' exist on an interval $I = (a, b)$.

Theorem 3. *If $f'(x) > 0$ for all $x \in I$, then f is increasing on I.*

Proof. Consider $x_1, x_2 \in I$, $x_1 < x_2$. We would like to show that $f(x_1) < f(x_2)$. Lagrange theorem implies the existence of a point $c \in (x_1, x_2)$, such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1). \tag{6.3}$$

The assumptions $x_2 > x_1$ and $f'(\boldsymbol{c}) > 0$ imply that the right side of the equation (6.3) is positive, thus $f(x_2) > f(x_1)$.

Corollary. *If f is decreasing or non-increasing on an interval I and has a derivative on I, then $f'(x) \leq 0$.*

Analogously:

Theorem 4. *If $f'(x) \geq 0$ for all $x \in I$, then f is non-decreasing on the interval I.*

Theorem 5. *If $f'(x) < 0$ for all $x \in I$, then f is decreasing on the interval I.*

Proof. Let $x_1, x_2 \in I$, $x_1 < x_2$. We have to prove that $f(x_1) > f(x_2)$. The Lagrange theorem implies the existence of $c \in (x_1, x_2)$, such that (6.3). Since $x_2 > x_1$ and $f'(c) < 0$, the right side of (6.3) is negative, thus $f(x_2) < f(x_1)$.

Corollary. *If f is increasing or non-decreasing on an interval I and its derivative exists on I, then $f'(x) \geq 0$.*

Theorem 6. *If $f'(x) \leq 0$ for all $x \in I$, then f is non-increasing on the interval I.*

☛ ***Example 3.***

Determine the intervals of monotonicity of the function

$$f(x) = 12x - 2x^2\,.$$

Solution.

$$f'(x) = 12 - 4x = 4(3 - x) = 0 \qquad \text{for} \quad x = 3\,;$$

$$f(x) > 0 \;\text{ for }\; x < 3; \qquad f(x) < 0 \;\text{ for }\; x > 3.$$

The function is increasing on $(-\infty, 3)$, decreasing on $(3, \infty)$.

Local (relative) extremes

Definition 4. A function f has **a local (relative) maximum**, resp. **local (relative) minimum** at $x_0 \in D_f$ if and only if there exists a punctured neighbourhood $P(x_0)$ such that $f(x) \leq f(x_0)$, resp. $f(x) \geq f(x_0)$, for all $x \in P(x_0)$.

If we replace unstrict inequalities by strict ones, we speak on **strict local maximum**, resp. **strict local minimum**.

Local maximas a minimas are also called **local extremes**, strict maximas and minimas are called **strict local extremes** of a function f.

Theorem 7. *Let $x_0 \in D_f$ be not a boundary point of the domain D_f of a function f. If $f'(a) \neq 0$, then f does not have an extreme at x_0.*

Proof. Let $f'(x_0) = A \neq 0$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$A - \varepsilon < \frac{f(x) - f(a)}{x - a} < A + \varepsilon\,.$$

for all $x \in P_\delta(x_0)$. Suppose for example that $A > 0$ and consider $\varepsilon = \dfrac{A}{2}$. Then

$$0 < \frac{A}{2} < \frac{f(x) - f(a)}{x - a}\,.$$

for all $x \in P_\delta(x_0)$. Further, $f(x) > f(a)$ for $x > a$ and $f(x) < f(a)$ for $x < a$. The function f therefore does not have a local extreme at x_0.

Similarly for $A < 0$. $\square$

Remark. The fact that $f'(x_0) > 0$ or $f'(x_0) < 0$ does not imply that f is increasing or decreasing on some neighbourhood of x_0.

☛ ***Example 4.***

Consider a function $f(x) = x + \pi x^2 \sin\dfrac{1}{x}$ pro $x \neq 0\,;\ f(0) = 0$. Obviously, $f'(0) = 1$, but the function is not increasing on any neighbourhood of $x = 0$, because for sufficiently high $n \in \mathbb{N}$ it is

$$f\left(\frac{1}{(2n+1/2)\pi}\right) > f\left(\frac{1}{(2n-1/2)\pi}\right) .$$

Theorem 8. *Let f be a differentiable function, let $f'(x_0) = 0$.*
If there exists $P_\delta(x_0)$ such that

$f'(x) > 0$ *for* $x < x_0$ *and* $f'(x) < 0$ *for* $x > x_0$, $x \in P_\delta(x_0)$,

then f has a strict local maximum at x_0.

If there exists $P_\delta(x_0)$ such that

$f'(x) < 0$ *for* $x < x_0$ *and* $f'(x) > 0$ *for* $x > x_0$, $x \in P_\delta(x_0)$,

then f has a strict local minimum at x_0.

If there exists $P_\delta(x_0)$ such that $f'(x) < 0$ or $f'(x) > 0$ for all $x \in P_\delta(x_0)$, then f does not have a local extreme at x_0.

☛ ***Example 5.***

Find local extremes of

$$f(x) = 12x - 2x^2 \,.$$

Solution.

$$f'(x) = 12 - 4x = 4(3 - x) = 0 \qquad \text{for} \quad x = 3\,;$$

$$f(x) > 0 \ \text{ for } \ x < 3; \qquad f(x) < 0 \ \text{ for } \ x > 3.$$

f is increasing on $(-\infty, 3)$, decreasing on $(3, \infty)$, thus it has a strict local maximum at 3, namely 18 .

Global extremes

Sometimes we need to find **global extremes on a compact, i.e., bounded and closed set** M*. The Weierstrass theorem implies that if a function* $f(x)$ *is continuous, then there exist poitsin* M *in which* $f(x)$ *attains its maximal and minimal value. Obviously, if* x *is not a boundary point of* M *and* $f'(x) \neq 0$, *then* f *does not have a global extreme at* x*. It is therefore sufficient to investigate the remaining points of* M :

- *boundary points of* M
- *points where the derivative is equal to zero*
- *points where the derivative does not exist*

It there exist only a finite number of such points, it is sufficient to compare their function values and select maximum and minimum.

☛ ***Example 6.***

Find global extremes of a function

$$f(x) = |x^3 - 3x|\,, x \in \langle -2\sqrt{3}, \sqrt{3} \rangle\,.$$

Solution. f is continuous on a compact interval $\langle -2\sqrt{3}, \sqrt{3} \rangle$, global minimum and maximum exist.

Candidates:

- Boundary points, i.e., $-2\sqrt{3}$ a $\sqrt{3}$
- $f'(x)$ does not exist for $-\sqrt{3}$, 0 and $\sqrt{3}$
- $f'(x) = 0$ for -1, 1

Now it is sufficient to find and compare function values:

$$f(-2\sqrt{3}) = 18\sqrt{3}\,, \quad f(\sqrt{3}) = f(-\sqrt{3}) = f(0) = 0\,,$$

$$f(-1) = f(1) = 2\,.$$

f attains a global maximum $18\sqrt{3}$ at $-2\sqrt{3}$ and global minimum 0 at $-\sqrt{3}$, $\sqrt{3}$ and 0.

Convex and concave functions, inflex points

Definition 5. Let $f'(x_0)$ exist. We say that f is **convex**, resp. **concave at** x_0 if and only if there exists $\mathrm{U}(x_0;\delta)$ such that the graph of f lies above, resp. below a tangent x_0 for all $x \in \mathrm{U}(x_0;\delta)$.

A function f is called **convex**, resp. **concave on an iterval** (a, b) if and only if it is convex, resp. concave at each point of (a, b).

A point $x_0 \in \mathrm{D}_f$ is called **an inflex point** of f if and only if there exists a tangent to its graph at $\big(x_0, f(x_0)\big)$ such that f changes from convex to concave or vice versa at this point.

Theorem 9. *Let* $f : (a,b) \to \mathbb{R}$ *have a derivative on* (a,b). *Then*

(i) *If* f' *is increasing on* (a,b), *then* f *is convex on* (a,b).

(ii) *If* f' *is decreasing on* (a,b), *then* f *is concave on* (a,b).

(iii) *If* f' *has a local extreme at* $x_0 \in (a,b)$, *then* x_0 *is an inflex point of* f .

Thus:

Theorem 10. *Let* $f : (a,b) \to \mathbb{R}$ *have a second derivative on* (a,b). *Then:*

(i) *If* $f''(x) > 0$ *for all* $x \in (a,b)$, *then* f *is convex on* (a,b).

(ii) *If* $f''(x) < 0$ *for all* $x \in (a,b)$, *then* f *concave on* (a,b).

(iii) *If* $f''(x_0) = 0$ *and* f'' *changes a sign at* $x_0 \in (a,b)$, *then* f *hax an inflex point at* x_0.