CHAPTER 9

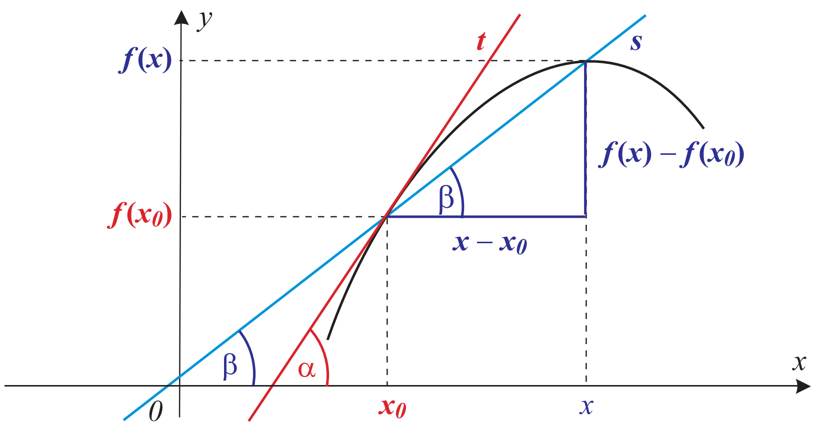
DERIVATIVES OF FUNCTIONS OF MORE VARIABLES

Derivative of a function

Recall that for a function of one variable, the derivative at an interior point x_0 of its domain was defined as the limit

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0},$$

10 pt



Derivative with respect to a vector

Definition 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables $\boldsymbol{a} \in D_f$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ a vector in $\mathbb{R}^n$. Let $\boldsymbol{a}$ be an accumulation point of a set $M = \{\boldsymbol{x}\,;\ \boldsymbol{x} = \boldsymbol{a} + \boldsymbol{v}t\,,\ t \in \mathbb{R}\} \cap D_f$ and $F(t) = f(\boldsymbol{a} + \boldsymbol{v}t)$. If the derivative $F'(0)$ exists, it is called **the derivative of $f(\boldsymbol{x})$ at the point $\boldsymbol{a}$ with respect to the vector $\boldsymbol{v}$.** It is also denoted by $\dfrac{\partial f}{\partial \boldsymbol{v}}$, $f'_{\boldsymbol{v}}$ etc.

The derivative with respect to a $\boldsymbol{v}$ for which $\|\boldsymbol{v}\| = 1$ is called **the derivative in the direction $\boldsymbol{v}$**.

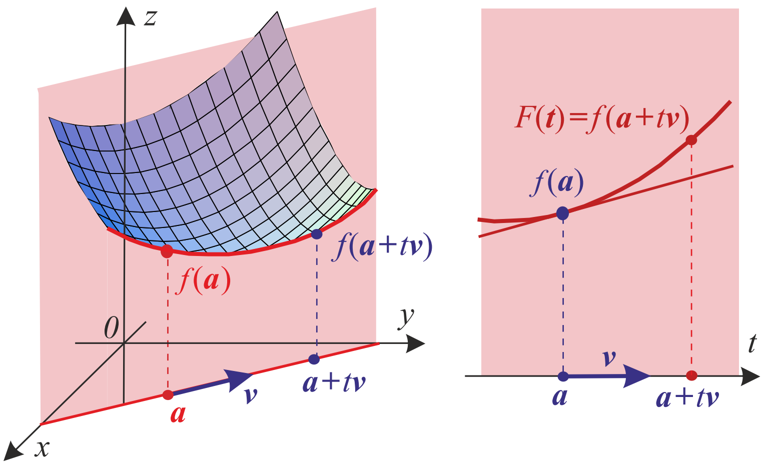
The derivative $f_{\boldsymbol{e}_i}$ with respect to a vector $\boldsymbol{v} = \boldsymbol{e}_i$ (a unit vector in the direction of the axis x_i) is called **the partial derivative with respect to the variable** x_i and denoted $\dfrac{\partial f}{\partial x_i}$ or f'_i.

The derivative of $f(x)$ with respect to a vector v at a point a

is the derivative of a function $F(t) = f\big(\boldsymbol{a} + \boldsymbol{v}t\big)$ of one variable t at 0 :

$$f'_{\boldsymbol{v}}(\boldsymbol{a}) = F'(0) =$$

$$\lim_{t\to 0} \frac{f(\boldsymbol{a} + \boldsymbol{v}t) - f(\boldsymbol{a})}{t} = = \lim_{t\to 0} \frac{f\big(a_1 + v_1 t, a_2 + v_2 t, \ldots, a_n + v_n t\big) - f\big(a_1, a_2,}{t}$$

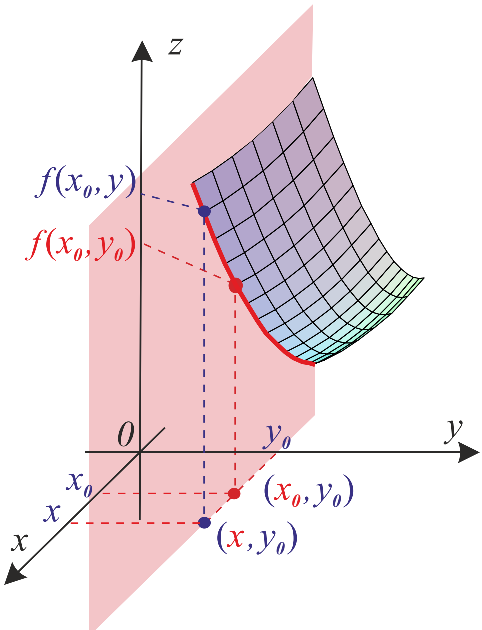


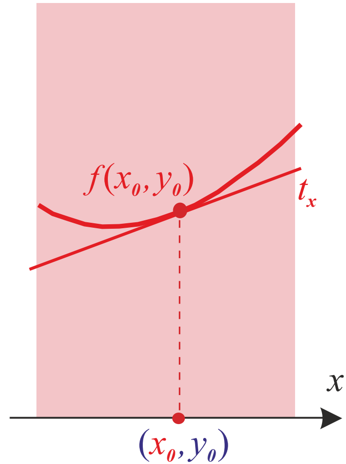
Special case: **the partial derivative of $f(\boldsymbol{x})$ with respect to a variable x_i at a point $\boldsymbol{a}$** is the derivative of $f(\boldsymbol{x})$ with respect to a unit vector in the direction of a coordinate x_i, i.e., with respect to a vector $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$.

$$\frac{\partial f}{\partial x_i}(\boldsymbol{a}) = \lim_{t \to 0} \frac{f\big(a_1, \ldots, a_{i-1}, a_i + t, a_{i+1}, \ldots, a_n\big) - f\big(a_1, \ldots, a_i, \ldots, a_n\big)}{t}.$$

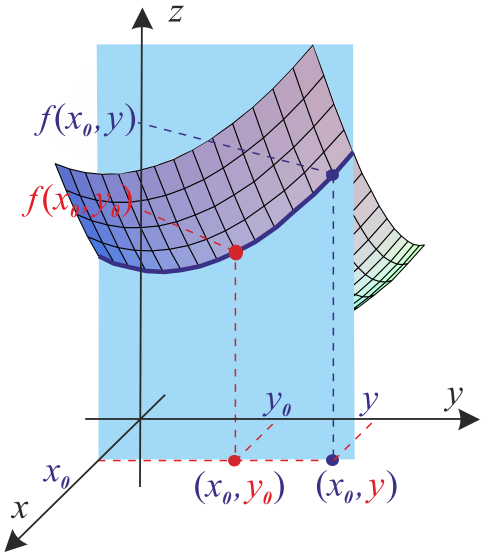
It means, that the partial derivative of $f(\boldsymbol{x})$ with respect to a variable x_i can be found in such a way that all other variables are considered as constants and the function is differentiated as a function of a unique variable x_i.

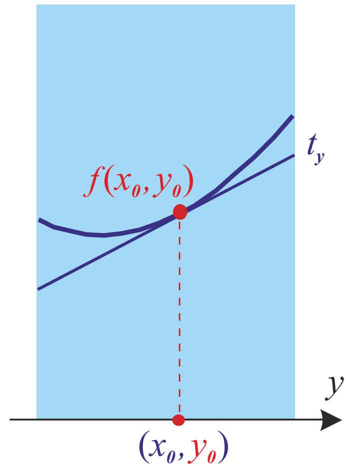
For a function $f(x, y)$ of two variables:





$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{t \to 0} \frac{f\left(x_0 + t, y_0\right) - f\left(x_0, y_0\right)}{t} = \lim_{x \to x_0} \frac{f\left(x, y_0\right) - f\left(x_0, y_0\right)}{x - x_0}.$$





$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{t\to 0} \frac{f\left(x_0, y_0 + t\right) - f\left(x_0, y_0\right)}{t} = \lim_{y\to y_0} \frac{f\left(x_0, y\right) - f\left(x_0, y_0\right)}{y - y_0}\,.$$

Since the derivatives with respect to a vector are derivatives of functions of one variables, the well-known relations hold:

Theorem 1. *Let* $f(\boldsymbol{x})$ *and* $g(\boldsymbol{x})$ *have derivatives with respect to a vector* $\boldsymbol{v}$ *at* $\boldsymbol{a}$, *let* $\alpha \in \mathbb{R}$ *be a constant. Then*

$$\begin{aligned}
\big(\alpha f\big)'_{\boldsymbol{v}}(\boldsymbol{a}) &= \alpha f'_{\boldsymbol{v}}(\boldsymbol{a})\,, \\
\big(f+g\big)'_{\boldsymbol{v}}(\boldsymbol{a}) &= f'_{\boldsymbol{v}}(\boldsymbol{a}) + g'_{\boldsymbol{v}}(\boldsymbol{a})\,, \\
\big(fg\big)'_{\boldsymbol{v}}(\boldsymbol{a}) &= f'_{\boldsymbol{v}}(\boldsymbol{a})g(\boldsymbol{a}) + f(\boldsymbol{a})g'_{\boldsymbol{v}}(\boldsymbol{a})
\end{aligned}$$

and for $g(\boldsymbol{a}) \neq 0$:

$$\left(\frac{f}{g}\right)'_{\boldsymbol{v}}(\boldsymbol{a}) = \frac{f'_{\boldsymbol{v}}(\boldsymbol{a})g(\boldsymbol{a}) - f(\boldsymbol{a})g'_{\boldsymbol{v}}(\boldsymbol{a})}{g^2(\boldsymbol{a})}\,.$$

Theorem 2. *For every vector* $\boldsymbol{v}$ *and every constant* $k \in \mathbb{R}$:

$$f'_{k\boldsymbol{v}}(\boldsymbol{x}) = k f'_{\boldsymbol{v}}(\boldsymbol{x})$$

Remark.

In general, the equality $f'_{(\boldsymbol{v}_1+\boldsymbol{v}_2)} = f'_{\boldsymbol{v}_1} + f'_{\boldsymbol{v}_2}$ does not hold.

Definition 2. Let $f\colon \mathrm{D}_f \subset \mathbb{R}^n \to \mathbb{R}^1$ have partial derivatives with respect to all variables, let these derivatives be continuous at a point $\boldsymbol{a} \in \mathrm{D}_f^\circ$. Then the linear function

$$\mathrm{d}f(\boldsymbol{a},\boldsymbol{h}) = \operatorname{grad} f(\boldsymbol{a}) \bullet \boldsymbol{h} \equiv$$

$$\equiv \frac{\partial f}{\partial x_1}(\mathbf{a})h_1 + \frac{\partial f}{\partial x_2}(\boldsymbol{a})h_2 + \cdots + \frac{\partial f}{\partial x_n}(\boldsymbol{a})h_n, \qquad \boldsymbol{h} \in \mathbb{R}^n,$$

is called **(total) differential of the function f at the point $\boldsymbol{a}$.**

The differential of a function f at $\boldsymbol{a}$ is often denoted as

$$\mathrm{d}f(\boldsymbol{a},\boldsymbol{x}-\boldsymbol{a}) = \operatorname{grad} f(\boldsymbol{a}) \bullet (\boldsymbol{x}-\boldsymbol{a}) \equiv \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(\boldsymbol{a})(x_k - a_k).$$

Another notation:

$$\mathbf{d}\boldsymbol{x} = (\,\mathrm{d}x_1,\,\mathrm{d}x_2,\ldots,\,\mathrm{d}x_n) = \boldsymbol{x} - \boldsymbol{a} = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$$

$$\mathrm{d}f(\boldsymbol{a},\,\mathbf{d}\boldsymbol{x}) = \operatorname{grad} f(\boldsymbol{a}) \bullet \mathbf{d}\boldsymbol{x} \equiv \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(\boldsymbol{a})\,\mathrm{d}x_k.$$

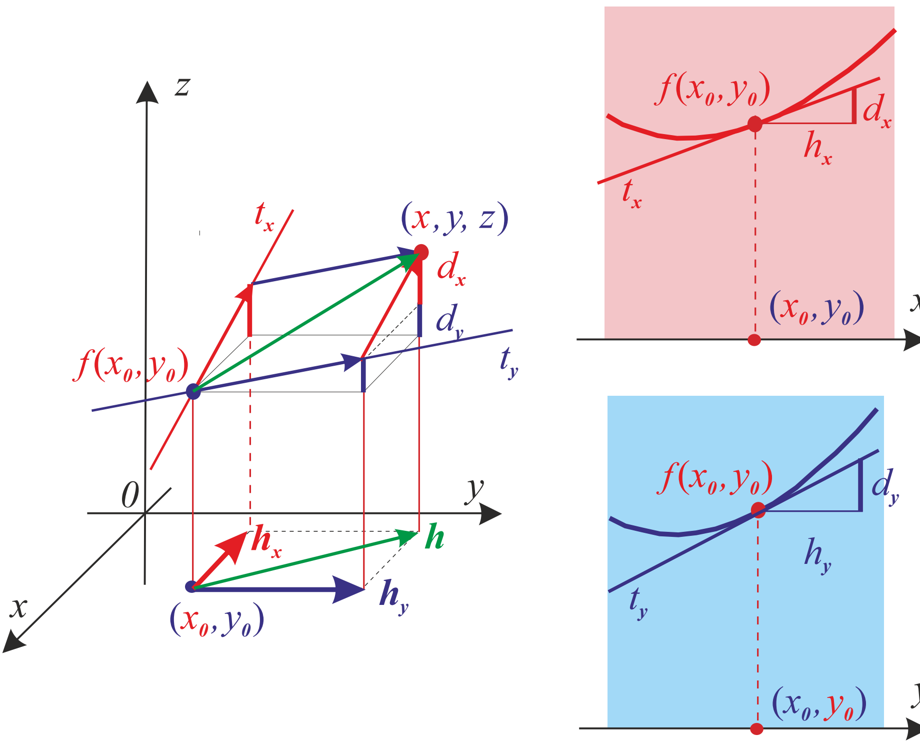
For a function of two variables:

$$\mathrm{d}f((x,y),(\,\mathrm{d}x,\,\mathrm{d}y)) = \frac{\partial f}{\partial x}(x,y)\,\mathrm{d}x + \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}y$$

For a function of three variables:

$$\mathrm{d}f((x,y,z),(\,\mathrm{d}x,\,\mathrm{d}y,\,\mathrm{d}z)) =$$

$$= \frac{\partial f}{\partial x}(x,y,z)\,\mathrm{d}x + \frac{\partial f}{\partial y}(x,y,z)\,\mathrm{d}y + \frac{\partial f}{\partial z}(x,y,z)\,\mathrm{d}z.$$


z
t_x
(x, y, z)
d_x
d_y
$f(x_0, y_0)$
t_y
0
y
h_x
h
x
h_y
(x_0, y_0)
$f(x_0, y_0)$
d_x
h_x
t_x
(x_0, y_0)
x
$f(x_0, y_0)$
d_y
h_y
t_y
(x_0, y_0)
y

Definition 3. Let $f(\boldsymbol{x})$ have a differential at a point $\boldsymbol{a}$. Then the vector

$$\operatorname{grad} f(\boldsymbol{a}) = \big(f_1'(\boldsymbol{a}), f_2'(\boldsymbol{a}), \ldots, f_n'(\boldsymbol{a})\big)$$

is called **gradient of $f(\boldsymbol{x})$ at $\boldsymbol{a}$.**

If all partial derivatives are continuous at $\boldsymbol{a}$, then

$$f_{\boldsymbol{v}}'(\boldsymbol{a}) = \boldsymbol{v} \cdot \operatorname{grad} f(\boldsymbol{a}) = \mathrm{d}f(\boldsymbol{a}, \boldsymbol{v})\,.$$

If the function $f\colon \mathrm{D}_f \subset \mathbb{R}^2 \to \mathbb{R}^1$ has a differential at $\boldsymbol{a} = (x_0, y_0) \in \mathrm{D}_f^\circ$ then the plane in $\mathbb{R}^3$ given by the equation

$$z - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0),$$

for $(x, y) \in \mathbb{R}^2$ is called **a tangent plane to the graph of the function f at the point $(\mathrm{a}, f(\boldsymbol{a}))$. The normal vector:**

$$\boldsymbol{n} = \left(\frac{\partial f}{\partial x}(\boldsymbol{a}), \frac{\partial f}{\partial y}(\boldsymbol{a}), -1\right)$$