EXERCISES 7 – Derivatives, further exercises

1. **Tangents, normals**

a) Find the equation of a tangent to the graph of the function $f(x) = x^3 + x - 2$, that is parallel to the line $y = 4x - 1$. $\Big[4x - y - 4 = 0 \text{ or } 4x - y = 0\,.\Big]$

b) Find the equation of a tangent to the graph of the function $f(x) = x^3 + 3x^2 - 5$, that is perpendicular to the line $2x - 6y + 1 = 0$. $\Big[3x + y + 6 = 0\,.\Big]$

c) Find the equation of a tangent to the graph of the function $f(x) = x \ln x$, that is parallel to the line $2x - 2y + 3 = 0$. $\Big[x - y - 3e^{-2} = 0\,.\Big]$

d) Find the equations of tangents to the hyperbola $7x^2 - 2y^2 = 14$, that are perpendicular to the line $2x + 4y - 3 = 0$. $\Big[2x - y - 1 = 0 \text{ or } 2x - y + 1 = 0\,.\Big]$

e) Find the equation of tangents to the parabola $y = \frac{1}{2}\,x^2 + 2$, that contain the point $A = [1; 2]$. $\Big[y = 2 \text{ or } y = 2x\,.\Big]$

f) Find the equation of a tangent to the graph of the function $f(x) = 2x^2 - 1$, that contains the point $A = [2; 5]$. $\Big[4x - y = 3 \text{ or } 12x - y = 19\,.\Big]$

g) Find the equation of tangents to the graph of the function $f(x) = \dfrac{1}{x}$, that contain the point $A = [-1; 3]$. $\Big[x + y - 2 = 0 \text{ or } 9x + y + 6 = 0\,.\Big]$

h) Find the domain of the function $f(x) = \ln \dfrac{x}{\sqrt{1 - x^2}}$ and determine the equations of tangents to the graph of this function at the points $\left[\frac{1}{2}; ?\right]$ a $\left[-\frac{1}{2}; ?\right]$.

$\Big[D_f = (0, 1)$; tangent at $\left[\frac{1}{2}, -\ln\sqrt{3}\right]$ je $8x - 3y - 4 - 3\ln\sqrt{3} = 0$; $x = -\frac{1}{2} \notin D_f\,.\Big]$

i) Find the domain of the function $f(x) = \big(\cos x\big)^{2x} + \arcsin \frac{1}{3}x$ and determine the equations of tangents to the graph of this function at the points $\big[0; ?\big]$ a $\big[2\pi; ?\big]$.

$\Big[D_f = \big(-\frac{1}{2}\,\pi; \frac{1}{2}\,\pi\big)$; the tangent at $[0; 1]$ is $x - 3y + 3 = 0$; $x = 2\pi \notin D_f\,.\Big]$

j) Find the domain of the function $f(x) = \ln \dfrac{x}{\sqrt{1 - x^2}}$ and determine the equations of tangents to the graph of this function at the points $\left[\frac{1}{2}; ?\right]$ a $\left[-\frac{1}{2}; ?\right]$.

$\Big[D_f = (0, 1)$; the tangent at $\left[\frac{1}{2}, -\ln\sqrt{3}\right]$: $8x - 3y - 4 - 3\ln\sqrt{3} = 0$; $x = -\frac{1}{2} \notin D_f\,.\Big]$

k) Find the domain of the function $f(x) = \ln \dfrac{3 - x^2}{3x^3}$ and determine the equations of tangents to the graph of this functions at the points $\big[2; ?\big]$ a $\big[1; ?\big]$.

$$\left[\begin{array}{l} D_f = \big(-\infty, -\sqrt{3}\big) \cup \big(0, \sqrt{3}\big)\,; \\ x = 2 \notin D_f\,; \quad \text{the tangent at } \left[1, \ln\frac{2}{3}\right] \text{ is } 4x + y - 4 - \ln\frac{2}{3} = 0. \end{array}\right]$$

l) Find the domain of the function $f(x) = \big(e^2 - x^2\big)^{\sin x}$ and determine the equations of tangents to the graph of this functions at the points $\big[3; ?\big]$ a $\big[0; ?\big]$.

$\Big[D_f = (-e, e)$; $x = 3 \notin D_f$; the tangent at $[0; 1]$ is $2x - y + 1 = 0\,.\Big]$

m) Find the equation of a tangent to the graph of the function $f(x) = (2e^x - \cos x)^{2x+1}$ at $\big[0; f(0)\big]$. $\Big[2x - y + 1 = 0\,.\Big]$

n) Find the equation of a tangent to the graph of the function $f(x) = (\tan x)^x$ at $\left[\frac{1}{4}\pi; f(\frac{1}{4}\pi)\right]$. $\left[\pi x - 2y + 2 - \frac{1}{4}\pi^2 = 0\,.\right]$

o) Find the equation of a tangent to the graph of the function $f(x) = (\cos x + \sin 2x)^{\cos x}$ at $\left[0; f(0)\right]$. $\left[2x - y + 1 = 0\,.\right]$

p) Find the equation of a tangent to the graph of the function $f(x) = (1 + \sin 2x)^{\cos x}$ at $\left[0; f(0)\right]$. $\left[2x - y + 1 = 0\,.\right]$

q) Find the equation of a tangent to the graph of the function $f(x) = (1 + 2\ln x)^x$ at $\left[1; f(1)\right]$. $\left[2x - y - 1 = 0\,.\right]$

r) Find the equation of a tangent to the graph of the function $f(x) = (1 + \tan x)^{x-2}$ at $\left[0; f(0)\right]$. $\left[2x + y - 1 = 0\,.\right]$

s) Find the equation of a tangent to the graph of the function $f(x) = (1 + \cos x)^{2\sin x}$ at $\left[\frac{1}{2}\pi; f(\frac{1}{2}\pi)\right]$. $\left[2x + y - \pi - 1 = 0\,.\right]$

t) Find the equation of a tangent to the graph of the function $f(x) = (1 - 3\sin x)^{\cos x}$ at $\left[0; f(0)\right]$. $\left[3x + y - 1 = 0\,.\right]$

u) Find the equation of a tangent to the graph of the function $f(x) = (\cos x + 2\sin x)^{x+1}$ at $\left[0; f(0)\right]$. $\left[2x - y + 1 = 0\,.\right]$

v) Find the equation of a tangent to the graph of the function $f(x) = (1 + 2\sin x)^{\cos x}$ at $\left[0; f(0)\right]$. $\left[2x - y + 1 = 0\,.\right]$

w) Find the equation of a tangent to the graph of the function $f(x) = (1 - 3x)^{e^x}$ at $\left[0; f(0)\right]$. $\left[3x + y - 1 = 0\,.\right]$

x) Find the equation of a tangent to the graph of the function $f(x) = x^{x(x+1)}$ at $\left[1; f(1)\right]$. $\left[2x - y - 1 = 0\,.\right]$

y) Find the equation of a tangent to the graph of the function $f(x) = \sqrt[2x]{x}$ at $\left[1; f(1)\right]$. $\left[x - 2y + 1 = 0\,.\right]$

z) Find the equation of a tangent to the graph of the function $f(x) = (1 - 2x)^{\cos x}$ at $\left[0; f(0)\right]$. $\left[2x + y - 1 = 0\,.\right]$

2. **Tangents, normals**

a) Find the domain of the function $f(x) = \left(\cos x\right)^{\cosh x} + \arctan 3x$ and determine the equations of tangents to the graph of this functions at the points $\left[0; ?\right]$ a $\left[\frac{2}{3}\pi; ?\right]$.

$$\left[\begin{array}{l} D_f = \bigcup\limits_{k\in\mathbb{Z}} \left(-\frac{1}{2}\pi + 2k\pi, \frac{1}{2}\pi + 2k\pi\right); \\ \text{the tangent at } [0;1] \text{ is } 3x - y + 1 = 0;\ x = \frac{2}{3}\pi \notin D_f. \end{array}\right]$$

b) Find the domain of the function $f(x) = \left(\cos x\right)^{2x} + \arccos \frac{1}{2}x$ and determine the equations of normals to the graph of this functions at the points $\left[0; ?\right]$ a $\left[\pi; ?\right]$.

$$\left[\begin{array}{l} D_f = \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right); \\ \text{the normal at the point } \left[0; 1 + \frac{1}{2}\pi\right] \text{ is } 4x - 2y + 2 + \pi = 0;\ x = \pi \notin D_f. \end{array}\right]$$

c) Find the equation of a normal to the graph of the function $f(x) = (\sin x + 2\cos x)^x$ at $\left[\frac{1}{2}\pi; f(\frac{1}{2}\pi)\right]$. $\left[2x - 2\pi y + \pi = 0\,.\right]$

3. **Leibniz formula**

a) Find the 20th derivative of the function $f(x) = x^2 \sin x$.

$$\Big[f^{(20)}(x) = \big(x^2 - 380\big) \sin x - 40x \cos x \,. \Big]$$

b) Find the 10th derivative of the function $f(x) = (x+1) \cos 2x$.

$$\Big[f^{(10)}(x) = -2^{10} \big((x+1) \cos 2x + 5 \sin 2x \big) \,. \Big]$$

4. Find open intervals, in which the function $f(x)$ is constant, and determine the value of $f(x)$ in each of these intervals.

a) $f(x) = \arctan x + \arctan \frac{1-x}{1+x}$

$$\Big[f(x) = -\tfrac{3}{4}\,\pi \text{ in the interval } (-\infty, -1),\ f(x) = \tfrac{1}{4}\,\pi \text{ in the interval } (-1, +\infty) \,. \Big]$$

b) $f(x) = 2\arctan x + \arcsin \frac{2x}{1+x^2}$

$$\Big[f(x) = -\pi \text{ in the interval } (-\infty, -1),\ f(x) = \pi \text{ in the interval } (1, +\infty) \,. \Big]$$

c) $f(x) = 2\arctan x - \arccos \frac{1-x^2}{1+x^2}$

$$\Big[\ f(x) = 0 \text{ in the interval } (0, +\infty) \ . \Big]$$

5. Find the domain of the function $f(x) = \big(e^2 - x^2\big)^{\sin x} + \arctan \frac{1}{2}x$ and determine its differentials at the points $x_1 = 0$ a $x_2 = -4$.

$$\Big[D_f = (-e, e);\ \text{differential at } x = 0 \text{ is } \mathrm{d}f(0; h) = \tfrac{5}{2}\,h;\ x = -4 \notin D_f \,. \Big]$$

6. Find the limit

a) $\lim\limits_{x \to 0} \Big(\dfrac{1}{e^x - 1} - \dfrac{1}{x} \Big)$. $\Big[-\frac{1}{2} \,. \Big]$

b) $\lim\limits_{x \to 1} \Big(\dfrac{1}{x - 1} - \dfrac{1}{\ln x} \Big)$. $\Big[-\frac{1}{2} \,. \Big]$

c) $\lim\limits_{x \to 0} \dfrac{x - \sin x}{\arctan x - x}$. $\Big[-\frac{1}{2} \,. \Big]$

d) $\lim\limits_{x \to +\infty} x \Big(\pi - 2 \arccos \dfrac{1}{x} \Big)$. $\Big[2 \,. \Big]$

e) $\lim\limits_{x \to 1} \tan \dfrac{\pi x}{2} \cdot \ln x$. $\Big[-\frac{2}{\pi} \,. \Big]$

f) $\lim\limits_{x \to 1} \dfrac{x^x - x}{1 - x + \ln x}$. $\Big[-2 \,. \Big]$

g) $\lim\limits_{x \to +\infty} x(\pi - 2\arctan x)$. $\Big[2 \,. \Big]$

h) $\lim\limits_{x \to 1} \big(x^x - 1\big) \tan \dfrac{\pi x}{2}$. $\Big[-\frac{2}{\pi} \,. \Big]$

i) $\lim\limits_{x \to 0} \big(\cos x\big)^{1/x^2}$. $\Big[e^{-1/2} \,. \Big]$

j) $\lim\limits_{x \to 0} \dfrac{1}{x} \Big(\dfrac{1}{\sin x} - \dfrac{1}{x} \Big)$. $\Big[\frac{1}{6} \,. \Big]$

k) $\lim\limits_{x \to 0} \dfrac{x - \arcsin x}{x^3}$. $\Big[-\frac{1}{6} \,. \Big]$

l) $\lim\limits_{x \to 0} \left(\dfrac{2}{\pi} \arccos x \right)^{1/x}$. $\Big[-\frac{2}{\pi} \,. \Big]$

m) $\lim\limits_{x\to 0} \dfrac{(1+x)^x - 1}{x^2}$. $\Big[1\,.\Big]$

n) $\lim\limits_{x\to \pi/4} (\tan x)^{\tan 2x}$. $\Big[e^{-1}\,.\Big]$

o) $\lim\limits_{x\to \pi/2} \sin(\cos x)\cdot \tan x$. $\Big[1\,.\Big]$

p) $\lim\limits_{x\to 0} \sqrt[x]{1+\tan 2x}$. $\Big[e^2\,.\Big]$

q) $\lim\limits_{x\to 0} \sqrt[x]{\cos x - \sin x}$. $\Big[e^{-1}\,.\Big]$

r) $\lim\limits_{x\to 0} \dfrac{e^x - e^{\sin x}}{x^3}$. $\Big[\frac{1}{6}\,.\Big]$

s) $\lim\limits_{x\to 0} \dfrac{\ln(\cos 3x)}{\ln(\cos 2x)}$. $\Big[\frac{9}{4}\,.\Big]$

t) $\lim\limits_{x\to 0_+} (\sin 3x)^{1/\ln x}$. $\Big[e\,.\Big]$

u) $\lim\limits_{x\to +\infty} \left(\cos\dfrac{1}{x}\right)^{x^2}$. $\Big[e^{-1/2}\,.\Big]$

v) $\lim\limits_{x\to 0_+} \big(\cot x\big)^{\sin 2x}$. $\Big[1\,.\Big]$

7. Find all asymptotes to the graph of the function

a) $f(x) = \dfrac{\ln(x+2)}{x^2-1}$. $\Big[$vertical as. $x=-2$ and $x=0$, horizontal as. $y=0$ in $+\infty\,.\Big]$

b) $f(x) = \dfrac{\ln(4-x)}{9-x^2}$. $\Big[$vertical as. $x=4$ and $x=-3$, horizontal as. $y=0$ in $-\infty\,.\Big]$

c) $f(x) = \dfrac{e^{\sqrt{x}}-1}{x^2-x}$. $\Big[$vertical asymptotes $x=0$ and $x=1\,.\Big]$

d) $f(x) = \sqrt[3]{x^3+x^2}$. $\Big[$oblique asymptotes $y=x+\frac{1}{3}$ in $\pm\infty\,.\Big]$

e) $f(x) = \sqrt[3]{1-x^3}$. $\Big[$oblique asymptotes $y=-x$ in $\pm\infty\,.\Big]$

f) $f(x) = \dfrac{\arccos\sqrt{x}}{\sqrt{x-x^2}}$. $\Big[$vertical asymptote $x=0\,.\Big]$

g) $f(x) = x\ln\left(e+\dfrac{1}{x}\right)$. $\Big[$vertical as. $x=-e^{-1}$, oblique as. $y=x+e^{-1}$ in $\pm\infty\,.\Big]$

h) $f(x) = \dfrac{4x^2}{\sqrt{x^2-1}}$. $\Big[$vertical as. $x=\pm 1$, obl. as. $y=4x$ in $+\infty$ and $y=-4x$ in $-\infty\,.\Big]$

i) $f(x) = \dfrac{3x^3+2x^2-12x-8+e^{-x}}{x^2+x-6}$. $\Big[$vertical as. $x=2$ and $x=-3$, oblique as. $y=3x-1$ in $+\infty\,.\Big]$

j) $f(x) = \dfrac{2x^2+5x+3-2x\arctan x}{x+1}$.

$\Big[$ vertical asymptote $x=-1\,,$ oblique asymptotes $y=2x+3-\pi$ in $+\infty$ and $y=2x+3+\pi$ in $-\infty\,.\Big]$

k) $f(x) = \left(\dfrac{x+2}{x-2}\right)^x$. $\Big[$vertical asymptotes $x=\pm 2$, horizontal as. $y=e^4$ in $\pm\infty\,.\Big]$