Chapter 14 Multiple Integrals

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Taylor's Formula for $f(x,y)$ at the Point (a,b)

Theorem. Suppose $f(x,y)$ and its partial derivatives through order $n+1$ are continuous throughout an open rectangular region R centered at a point (a,b). Then, throughout R,

$$f(a+h,b+k) = \underbrace{f(a,b) + (hf_x + kf_y)|_{(a,b)}}_{\text{Linear or 1st order approximation}} + \cdots$$

$$f(a+h,b+k)$$
$$= \underbrace{f(a,b) + (hf_x + kf_y)|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})|_{(a,b)}}_{\text{2nd order approximation}} + \cdots$$

$$f(a+h,b+k)$$
$$= f(a,b) + (hf_x + kf_y)|_{(a,b)} + \tfrac{1}{2!}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})|_{(a,b)}$$
$$+ \tfrac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 2hk^2 f_{xyy} + k^3 f_{yyy})|_{(a,b)} + \cdots$$
$$+ \tfrac{1}{n!}\left(h\tfrac{\partial}{\partial x} + k\tfrac{\partial}{\partial y}\right)^n f|_{(a,b)} + \tfrac{1}{(n+1)!}\left(h\tfrac{\partial}{\partial x} + k\tfrac{\partial}{\partial y}\right)^{n+1} f|_{(a+ch,b+ck)}$$

for some $c \in (0,1)$.

Taylor's Theorem. Suppose $f(x,y)$ and its partial derivatives through order $n+1$ are continuous throughout an open rectangular region R centered at a point (a,b). Then, throughout R,

$$= f(a,b) + (hf_x + kf_y)|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})|_{(a,b)}$$
$$+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 2hk^2 f_{xyy} + k^3 f_{yyy})|_{(a,b)} + \cdots$$
$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f|_{(a,b)} + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f|_{(a+ch,b+ck)}$$

for some $c \in (0,1)$.

Remarks.

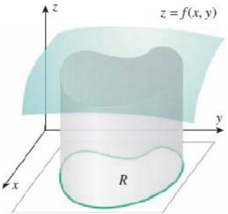
1. The proof just applies the chain rule and the trick of n-th Taylor polynomial to the function $g(t) = f(a+ht, b+kt)$ one variable.
2. If one have an estimate the last term (in blue), for example an upper bound, then we can estimate the given function by means of polynomials in 2 variables.
3. The theorem can be easily generalized to function of n variables for $n \geq 1$. Though this topics is not treated in this book, but its application is important in other courses, so we put the result in this notes for the sake of students

Definition. The definite integral of a function $f(x)$ of one variable defined on an interval $[a,b]$, is given by

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

This concept arose from the problem of finding areas under curves. Now we have similar problem if we replace the function of two variables.

Volume Problem. Given a function $f(x,y)$ of two variables that is continuous and nonnegative on a region R in the xy-plane, find the volume of the solid enclosed between the surface $z=f(x,y)$ and the region R.



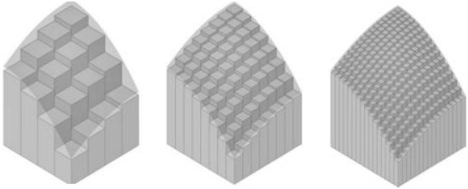
Volume and Double Integrals

Let $f(x,y)$ be a function of two variables defined over a rectangle $R = [a,b] \times [c,d]$. We would like to define the double integral of $f(x,y)$ over R as the algebraic volume of the solid under the graph of $z = f(x,y)$ over R.

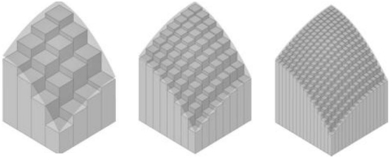
The idea is similar to the case of integral $\int_a^b f(x)dx$ in one variable case, in which we subdivide the interval into smaller subintervals with uniform width $\Delta x = \frac{b-a}{n}$, and then choose arbitrary points x_i in the subinterval I_i. Then we have the approximate Riemann sum

$$\sum_{i=1}^{n} f(x_i)\Delta x.$$

Of course, this sum depends on the choices of x_i and the subinterval. In fact, this idea can be implemented in 2-dimensional cases as well.



Let $R = [a, b] \times [c, d]$. and $f(x, y)$ be a function defined on R. We first subdivide the rectangle R into mn small rectangles R_{ij}, each having area ΔA, where $i = 1, \cdots, m$ and $j = 1, \cdots, n$. For each pair (i, j), pick an arbitrary point (x_{ij}, y_{ij}) inside R_{ij}. Use the value $f(x_{ij}, y_{ij})$ as the height of a rectangular solid erected over R_{ij}. Thus its volume is $f(x_{ij}, y_{ij})\Delta A$.



The sum of the volume of all these small rectangular solids approximates the volume of the solid under the graph of $z = f(x, y)$ over R. This sum $\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$ is called Riemann sum of f.

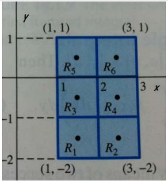
Definition The double integral of f over R is

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A, \text{ if this limit exists.}$$

Remarks. In general, it is very difficult to prove that the limit of Riemann sum converges, because of the choices of the height $f(x_{ij}, y_{ih})$ involved. The usual method is replace the height either the maximum and the minimum values of f within each smaller rectangles, and hence we obtain the upper and lower Riemann sums respectively.

Theorem. If $f(x,y)$ is continuous on a domain containing the rectangle R, then the double integral $\iint_R f(x,y)\, dA$ always exists.

Example. Approximate the value of the integral $\iint_R (4x^3 + 6xy^2)\, dA$ over the rectangle $R = [1, 3] \times [-2, 1]$, by means of the Riemann sums, with $\Delta x_i = 1$, and $\Delta y_j = 1$.



Solution. Partition the rectangle R into six 1×1 squares R_i with area $\Delta A_i = 1$ $(i = 1, \cdots, 6)$. Choose the center points (x_i^*, y_i^*) for each square as shown on the right.

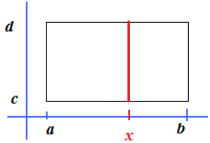
The desired Riemann sum is $\sum_{i=1}^{6} f(x_i^*, y_i^*)\Delta A_i$

$$\begin{aligned}
&= f(\tfrac{3}{2}, -\tfrac{3}{2}) \times 1 + f(\tfrac{5}{2}, -\tfrac{3}{2}) \times 1 + f(\tfrac{3}{2}, -\tfrac{1}{2}) \times 1 \\
&\quad + f(\tfrac{5}{2}, -\tfrac{1}{2}) \times 1 + f(\tfrac{3}{2}, \tfrac{1}{2}) \times 1 + f(\tfrac{5}{2}, \tfrac{1}{2}) \times 1 \\
&\quad = \frac{135}{4} + \frac{385}{4} + \frac{63}{4} + \frac{265}{4} + \frac{63}{4} + \frac{265}{4} = 294,
\end{aligned}$$

which is called the midpoint approximation of the integral $\iint_R (4x^3 + 6xy^2)\, dA$.

Iterated Integrals

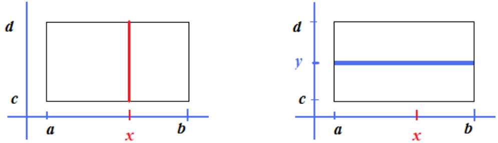
Let $f(x,y)$ be a function defined on $R = [a,b] \times [c,d]$. We write $\int_c^d f(x,y)\,dy$ to mean that x is regarded as a constant and $f(x,y)$ is integrated with respect to y from $y = c$ to $y = d$.



Therefore, the value of the integral $\int_c^d f(x,y)\,dy$ is a function of x, and we can integrate it with respect to x from $x = a$ to $x = b$. The resulting integral $\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$ is called an iterated integral.

Iterated Integrals

Let $f(x,y)$ be a function defined on $R = [a,b] \times [c,d]$. We write $\int_c^d f(x,y)\,dy$ to mean that x is regarded as a constant and $f(x,y)$ is integrated with respect to y from $y = c$ to $y = d$.



Therefore, the value of the integral $\int_c^d f(x,y)\,dy$ is a function of x, and we can integrate it with respect to x from $x = a$ to $x = b$. The resulting integral $\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$ is called an iterated integral. Similarly one can define the iterated integral $\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$.

Remark. We call the blue and red segments inside the region R the cross-sections of R cut by the line $y = y_0$ and $x = x_0$ respectively.

Example. Evaluate the iterated integrals

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$, (b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$.

Solution.

(a) $\int_0^3 \int_1^2 x^2 y \, dy dx = \int_0^3 \int_1^2 x^2 y \, dy dx = \int_0^3 x^2 \int_1^2 y \, dy dx ==$

$\int_0^3 x^2 \left[\frac{y^2}{2}\right]_{y=1}^{y=2} dx = \int_0^3 \frac{3x^2}{2} dx = \left[\frac{x^3}{2}\right]_{x=0}^{x=3} = \frac{27}{2}.$

(b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \left[\frac{x^3 y}{3}\right]_{x=0}^{x=3} dy = \int_1^2 9y dy = \left[\frac{9y^2}{2}\right]_{y=1}^{y=2} = \frac{27}{2}.$

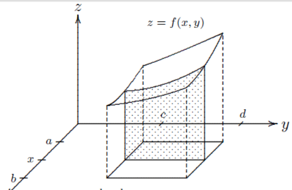
Fubini's Theorem for Rectangle case

Thoerem. If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$, then

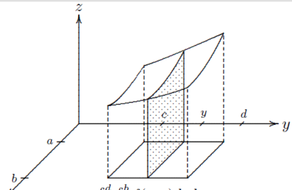
$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \iint_R f(x,y) dA = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

Example. $f(x,y)$ is a positive function defined on a rectangle $R = [a,b] \times [c,d]$. The volume V of the solid under the graph of $z = f(x,y)$ over R, is given by either one of the iterated integrals:

$$\int_a^b \int_c^d f(x,y)\, dy\, dx, \text{ or } \quad \int_c^d \int_a^b f(x,y)\, dx\, dy.$$



$\int_a^b \int_c^d f(x,y)dydx$



$\int_c^d \int_a^b f(x,y)dxdy$

Proof of Fubini Theorem for rectangle case

Thoerem. If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$, then
$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \iint_R f(x,y) dA = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

Proof. First partition $[a,b]$, and $[c,d]$ each into n equal subintervals, so that we have n^2 smaller rectangles of area $\Delta A = \Delta x \Delta y$. We construct a Riemann sum of f on R which is close to the iterated integral. For $i = 1, \cdots, n$, one can choose any point x_i^* in subinterval $[x_{i-1}, x_i]$. It follows from the mean value theorem of integral that there exists y_{ij}^* in subinterval $[y_{j-1}, y_i]$ with $\int_{y_{j-1}}^{y_j} f(x_i^*, y) dy = f(x_i^*, y_{ij}^*) \Delta y$. This produces a point (x_i^*, y_{ij}^*) in each rectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_i]$. Then the Riemann sum $\sum_{i,j=1}^n f(x_i^*, y_{ij}^*)\, \Delta A = \sum_{i=1}^n \left(\sum_{j=1}^n f(x_i^*, y_{ij}^*)\, \Delta y \right) \Delta x =$
$$\sum_{i=1}^n \left(\sum_{j=1}^n \int_{y_{i-1}}^{y_i} f(x_i^*, y) dy \right) \Delta x = \sum_{i=1}^n \left(\int_c^d f(x_i^*, y)\, dy \right) \Delta x = \sum_{i=1}^n A(x_i^*) \Delta x,$$
where $A(x) = \int_c^d f(x,y)\, dy$. The result follows as n tends to $+\infty$.

Fubini's Theorem for Rectangle case

Thoerem. If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$, then
$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \iint_R f(x,y) dA = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

More generally, this is true if f is bounded on R, f is discontinuous only at a finite number of smooth curves, and the iterated integrals exist.

Furthermore, the theorem is valid for a general closed and bounded region as discussed in the subsequent sections.

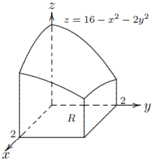
Example. Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$, $y = 2$, and the 3 coordinate planes.

Solution. Let $R = \{\ (x, y)\ 0 \le x \le 2,\ 0 \le y \le 2\ \}$, and one can rewrite the defining equation of the elliptic paraboloid as $z = 16 - x^2 - 2y^2$ where $(x, y) \in R$. Then the volume of S is given by the double integral

$$\iint_R f(x,y)\,dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\,dx\,dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - 2xy^2\right]_{x=0}^{x=2} dy$$

$$= \int_0^2 (\frac{88}{3} - 4y^2)\,dy = \frac{88 \times 2}{3} - \frac{4 \times 2^2}{3} = \frac{160}{3}.$$



Remark. In this problem, one has to decide which is the height function, just like $z = f(x, y)$ in the previous formulation. In some other cases, one can use $x = h(y, z)$ or $y = g(x, z)$ as the height function.

Proposition. (a) In general, if $f(x,y)=g(x)h(y)$, then

$$\iint_R f(x,y)dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right),$$

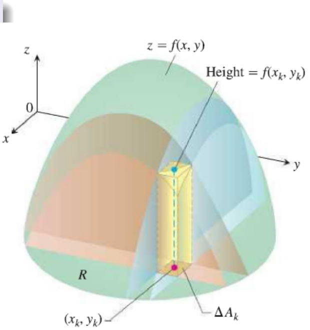
where $R=[a,b]\times[c,d]$ is a rectangle.
(b) The equation above does not hold if the region R is not a rectangle.

Solution. (a) $\iint_R f(x,y)dA = \int_a^b \int_c^d g(x)h(y)dy\,dx =$
$\int_a^b g(x) \int_c^d h(y)dy\,dx = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$

Remark. **Example.** $\exp(x^2+y^2)=e^{x^2+y^2}=e^{x^2}\cdot e^{y^2}$.
Counterexample. One can show that $\sin(x+y)$ can not be expressed as $g(x)\cdot h(y)$. However, one can relax the condition into sums of products of functions: $\sin(x+y)=\sin x\cdot\cos y+\cos x\cdot\sin y$.

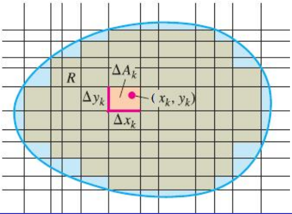
Double Integral over Non-Rectangular Region

Let $f(x,y)$ be a continuous function defined on a closed and bounded region D in $\mathbb{R}^2$. The double integral $\iint_D f(x,y)\, dA$ can be defined similarly as the limit of a Riemann sum $\sum_k f(x_k, y_k)\Delta A_k$, where small rectangle R_k with dimension $\Delta x_k \times \Delta y_k$ lies completely inside region R.



Volume $= \lim \sum f(x_k, y_k)\,\Delta A_k = \iint_R f(x, y)\, dA$

However, due to the irregular shapes of the region, we subdivide the region by rectangular grid, and then evaluate the volume of the rectangular solid with the base of smaller rectangles the completely lies inside the region R.



Example. If $R = \{ (x, y) \mid -1 \le x \le 1, \ -2 \le y \le 2 \}$, evaluate the double integral $\iint_R \sqrt{1 - x^2} dA$.

Solution. We can compute the integral by interpreting it as a volume of a solid body D. If we let $z = \sqrt{1 - x^2}$, then $z^2 = 1 - x^2$, i.e. $x^2 + z^2 = 1$, hence part of the boundary of D lies in the cylinder. , then and , so the given double integral represents the volume of the solid S that lies below the circular cylinder and above the rectangle R. The volume of S is the area of a semicircle with radius 1 times the length of the cylinder. Thus

$$\iint_R \sqrt{1 - x^2} dA = \text{Volume of} \quad S = \frac{1}{2}\pi \times 1^2 \times (2 - (-2)) = 2\pi.$$

Fubini theorem for non-rectangular region

Let R be a region in the xy-plane, and suppose there exist two continuous function $y_{\min}(x)$, $y_{\max}(x)$ defined on the interval $[a,b]$ such that $R = \{ (x,y) \mid a \leq x \leq b,\ y_{\min}(x) \leq y \leq y_{\max}(x) \}$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_{y_{\min}(x)}^{y_{\max}(x)} f(x,y)\, dy\, dx.$$

Using Vertical Cross-sections

In evaluating $\iint_R f(x,y)dA$, one can ideally use the iterated integral

$$\iint_R f(x,y)dA = \int_a^b \left(\int_?^* f(x,y)\, dy \right) dx,$$

the difficulties lies in determining upper and lower limits $*$, ? in the iterated integral.

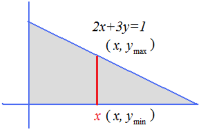
Vertical Cross-sections

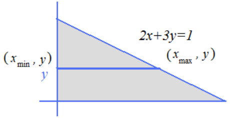
$$\iint_R f(x,y)dA = \int_a^b \left(\int_?^* f(x,y)\, dy \right) dx.$$

We propose the following steps:

1. Sketch and label the bounding curves, and determine the region R of integration in the double integral.
2. Project the region R onto one the coordinate axes, so that its shadow is an interval $[a,b]$ or union of intervals on the coordinate axis.
3. Choose any arbitrary point $P(x,0)$ or $P(0,y)$ in the shadow, draw a line ℓ through P perpendicular to the axis with shadow.
4. Ideally the line ℓ meets the boundary R at only two points $(x, y_{\max})$ and $(x, y_{\min})$. These two y's depends on x, and hence are functions of x, i.e. the ones determined by the boundary curves of R. In this case one can describe the region $R = \{\ (x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ y_{\min}(x) \leq y \leq y_{\max}(x)\ \}$.

Example. Sketch the region D bounded by the lines $x = 0$, $y = 0$ and $2x + 3y = 1$. and evaluate the double integral $\iint_D x\, dA$.



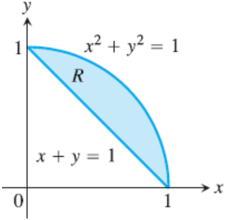


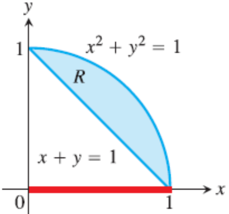
Solution. The region D is a triangle bounded by $x = 0$ and $y = 0$ and $\ell : 2x + 3y = 1$. First we determine the intersection points of these 3 lines. The two coordinates axes meet at $(0, 0)$; the line ℓ intersects the x- and y-axis at $(1/2, 0)$ and $(0, 1/3)$ respectively. Next we determine the order of integration in the iterated integral, for example $\int_a^b \left(\int \qquad dy\right) dx$. Then the region D has a shadow $\{\, x \mid 0 \leq x \leq 1/2 \,\}$ on x-axis. Then any vertical red line through a point $(x, 0)$ on the x-axis will intersect the region at two boundary points $(x, y_{\max})$ and $(x, y_{\min})$, where $y_{\min} = 0$ given by the x-axis, and $y_{\max} = y$ which satisfies $2x + 3y = 1$, i.e. $y_{\max} = \frac{1-2x}{3}$. So we have

$$\iint_D x dA = \int_0^{\frac{1}{2}} \int_0^{\frac{1-2x}{3}} x\, dy\, dx = \int_0^{\frac{1}{2}} \frac{x(1-2x)}{3}\, dx = \left[\frac{x^2}{6} - \frac{2x^3}{9}\right]_0^{\frac{1}{2}} = \frac{1}{72}.$$

Example. Let R be the region in first quadrant bounded by the two curves $x^2 + y^2 = 1$ and $x + y = 1$.

1. Sketch and label the bounding curves, and determine the region R of integration in the double integral.
2. Project the region R onto one the coordinate axes, so that its shadow is an interval $[a, b]$ or union of intervals on the coordinate axis.





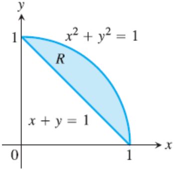
project R onto x-axis
with red shadow interval $[0,1]$

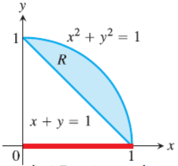
Example. Let R be the region in first quadrant bounded by the two curves $x^2 + y^2 = 1$ and $x + y = 1$.

Usually the intersection point of curves $C_1 : x + y = 1$ and $C_2 : x^2 + y^2 = 1$ gives some important information.
$1 = x^2 + y^2 = x^2 + (1 - x)^2 = 2x^2 - 2x + 1$, i.e. $0 = x(x - 1)$, and hence we know $(x, y) = (1, 0)$ and $(0, 1)$ are the common intersection.
Want to see (by mathematical means) the relative position of the curves $y = 1 - x$ and $y = \sqrt{1 - x^2}$ when x varies in the interval $[0, 1]$.
For $0 \le x \le 1$, we have $0 \le 1 - x \le 1 + x$, so

$$(1 - x)^2 \le (1 + x)(1 - x) = 1 - x^2,$$

it follows that $1 - x \le \sqrt{1 - x^2}$ for $0 \le x \le 1$, i.e. the line segment C_2 is below the circle C_2.

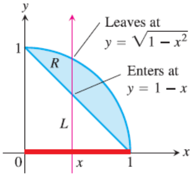




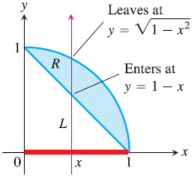
project R onto x-axis
with red shadow interval $[0,1]$

Example. Let R be the region in first quadrant bounded by the two curves $x^2+y^2=1$ and $x+y=1$. Rewrite the double integral $\iint_R f(x,y)dA$ in iterated integrals.

1. Choose any arbitrary point $P(x,0)$ or $P(0,y)$ in the shadow, draw a line ℓ through P perpendicular to the axis with shadow.
2. Ideally the line ℓ meets the boundary R at only two points $(x, y_{\max})$ and $(x, y_{\min})$. These two y's depends on x, and hence are functions of x, i.e. the ones determined by the boundary curves of R. Then the region $R = \{\ (x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ y_{\min}(x) \le y \le y_{\max}(x)\ \}$.



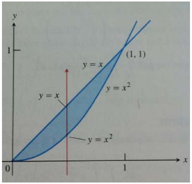
For any in the shadow interval, draw a vertical line cuts the boundary of R at two points with coordinates (x, $y_{\min}$) and (



Answer: $\iint_R f(x,y)dA = \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$

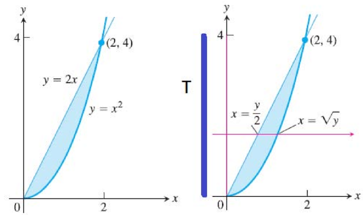
Proposition. The area of a region R in xy-plane is given by $\iint_R dA$.

Example. Find the area of the region R bounded by the curves $C_1 : y = x + 2$, and $C_2 : y = x^2$ in the first quadrant. Answer in $\int \int \cdots dydx$.



Solution. It is observed from the figure above that $R = \{ \ (x, y) \mid 0 \leq x \leq 1, \ x^2 \leq y \leq x \ \}$ ♡. We first find the intersection point (x, y) of C_1 and C_2. In this case, both equations $y = x$ and $y = x^2$ hold, i.e. $x = x^2$, and so $x = 0$ or $x = 0$. It follows that $(x, y) = (0, 0)$ or $(1, 1)$. As x runs along in the interval $[0, 1]$, i.e. $0 \leq x \leq 1$, one needs to decide which of these two curves $C_i \ (i = 1, 2)$ lies on top, while the other is at the bottom. One can compare the y-coordinates of the points (x, x) on C_1, and (x, x^2) on C_2, so it follows from $0 \leq x \leq 1$ that $x - x^2 = x(1 - x) \geq 0$. We know that C_1 lies above C_2 when $x \in [0, 1]$. So $R = \{ \ (x, y) \mid 0 \leq x \leq 1, \ x^2 \leq y \leq x \ \}$. The area of the region R is given by $\int_0^1 \int_{x^2}^{x} dydx = \int_0^1 (x - x^2) \, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$.

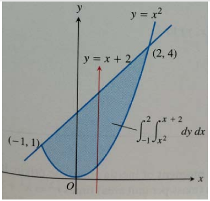
Example. Sketch the region of integration for the integral $\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$ and write an equivalent integral with the order of integration reversed.



Solution. It follows from the given iterated integral that the domain of integral $R = \{ (x, y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x \}$. In this case, the top and bottom curves intersect at $(0, 0)$ and $(2, 4)$.So the region R has a shadow $T = [0, 4]$ onto y-axis. For any point $P(y, 0)$ $(0 \le y \le 4)$ in the shadow T, the red line ℓ parallel to x-axis will meet the parabola $y = x^2$ at $(\sqrt{y}, y)$ and the straight line $y = x + 2$ at $(y - 2, y)$. It follows from $0 \le y \le 4$, that $y^2 \le 4y$, i.e. $y_{\min} = y/2 \le \sqrt{y} = y_{\max}$. Geometrically, ℓ meets the boundary curves at two points $(x_{\min}, y)$ and $(x_{\max}, y)$. One has $R = \{ (x, y) \mid 0 \le y \le 4,\ \frac{y}{2} \le x \le \sqrt{y} \}$. It follows that the

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy.$$

Example. Find the area of the region R bounded by the curves $C_1 : y = x + 2$, and $C_2 : y = x^2$ in the 1st quadrant. Answer in integral form $\int \int \cdots dydx$.

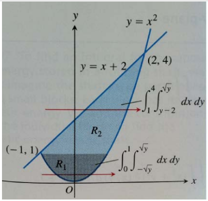


Solution. The region R can be described by means of vertical section as $\{ (x, y) \mid -1 \leq x \leq 2,\ x^2 \leq y \leq x + 2 \}$. It follows from the description of R that the area of the region R is given by

$$\int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx = \int_{-1}^{2} \Big[y \Big]_{x^2}^{x+2} dx = \int_{-1}^{2} (x + 2 - x^2)\,dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = \frac{9}{2}.$$

Example. Find the area of the region R bounded by the curves $C_1 : y = x$, and $C_2 : y = x^2$ in the 1st quadrant. Answer in integral form $\int\int \cdots dxdy$.

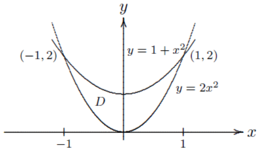


Solution. If one uses the cross sections parallel to x-axis, then we have the diagram above, in which the lower limits may be on C_1 if $y \geq 1$, and on C_2 if $0 \leq y \leq 1$. So one may divide the region R into two subregions R_1 and R_2 as above. In fact,

$R_1 = \{\ (x, y) \mid 0 \leq y \leq 1,\ -\sqrt{y} \leq x \leq \sqrt{y}\ \}$, and

$R_2 = \{\ (x, y) \mid 1 \leq y \leq 4,\ y - 2 \leq x \leq \sqrt{y}\ \}$. So it follows that area of R is $\iint_R 1\, dA = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dxdy + \int_1^4 \int_{y-2}^{\sqrt{y}} dxdy$.

Example. Evaluate $\iint_D (x+2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



Solution. We first determine the intersection of the curves $y = 2x^2$ and $y = 1 + x^2$ as follows. From $2x^2 = y = 1 + x^2$, we have $x^2 = 1$, and hence $x = \pm 1$. When x varies within the interval $[-1, 1]$, one has $x^2 \leq 1$, i.e. $2x^2 \leq 1 + x^2$. In particular, the curve $y = 2x^2$ is below the curve $y = 1 + x^2$. So $D = \{ (x, y) \mid 0 \leq x \leq 1,\ 2x^2 \leq y \leq 1 + x^2 \}$. And we have the figure on the right.

Rewrite the double integral as iterated integral,

$$\iint_D (x+2y)dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx = \int_{-1}^{1} \left[xy + y^2 \right]_{2x^2}^{1+x^2} = \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx = \frac{32}{15}.$$

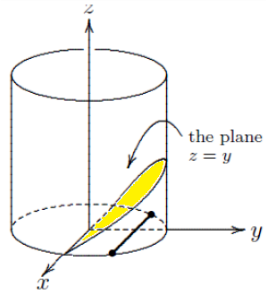
Exercise. Evaluate $\iint_D xy dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. Answer: 36.

Example. Evaluate the iterated integral $I = \int_0^1 \int_x^1 \sin(y^2)\, dydx$ by interchanging the order of integration.

Solution. Let D be the region The region of integration, then it follows from the upper and lower limits of the iterated integral, we have $D = \{ (x,y) \mid 0 \le x \le 1,\ x \le y \le 1 \}$, which is given by vertical cross-section. Hence D is the triangular region bounded by the lines $y = x,\ x = 0$ and $y = 1$. Rewrite D by means of horizontal cross-section, then $0 \le y \le 1$, and the bounding curve will be $x = 0$ on the left, and $x = y$ on the right of the region D. Hence, we have another description of $D = \{ (x,y) \mid 0 \le y \le 1,\ 0 \le x \le y \}$. By Fubini's theorem we have $I = \int_0^1 \int_x^1 \sin(y^2)\, dydx =$

$$\int_0^1 \int_0^y \sin(y^2)\, dxdy = \int_0^1 y \sin(y^2) dy = \left[-\frac{1}{2} \cos(y^2) \right]_0^1 = \frac{1 - \cos 1}{2}$$

Example. Find the volume of the solid S above the xy-plane and is bounded by the cylinder $x^2 + y^2 = 1$ and the plane $z = 0$ and $z = y$.

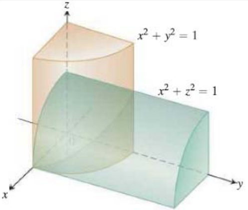


Solution. Since the plane $z = 0$ is the bottom of the solid, and the plane $z = y$ is the top face of the solid, we may use the function defining this plane $z = y$ as the height function of this solid. In other words, $f(x, y) = y$ is function appeared as integrand. Therefore, the volume of the solid can be computed by integrating this function f over the bottom face of the solid which is the semi-circular disk

$$D = \{\ (x, y) \mid x^2 + y^2 \le 1, y \ge 0\ \}$$
$$= \{\ (x, y) \mid 0 \le y \le 1,\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}\ \}.$$ The volume of the solid S is $\iint_D y\, dA = \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\, dx\, dy$

$$= \int_0^1 2y\sqrt{1-y^2}\, dy = \left[-\frac{2}{3}(1-y^2)^{3/2}\right]_0^1 = \frac{2}{3}.$$

Example. The region common to the interiors of the cylinders $x^2+y^2=1$ and $x^2+z^2=1$, one-eighth of which is shown in the accompanying figure.



Solution. We just calculate the volume of portion D of the region in 1st octant. One immediately recognizes the solid D has a top given by $x^2+z^2=1$, i.e. $z_{\max}(x)=\sqrt{1-x^2}$, and xy-plane as the bottom. Moreover, the shadow R of the solid D is a circle disk in the 1st quadrant, so $R=\{\ (x,y)\mid 0\le x\le 1,\ 0\le y\le \sqrt{1-x^2}\ \}$. The volume of D is given by $\iiint_D 1\,dV=\iint_R \int_0^{\sqrt{1-x^2}} dzdA$

$$=\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-x^2}\,dy\,dx=\int_0^1(1-x^2)dx=\left[x-\frac{x^3}{3}\right]_0^1=\frac{2}{3}.$$