Exercise 3 – Riemann integral in $\mathbb{R}^2$ and $\mathbb{R}^3$

Find the area of the region $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$y - 2x^2 - 4x \geq 0\,, \qquad 5x - y + 1 \geq 0\,.$$

$\left[\frac{9}{8}\right]$

Find the area of the region $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$8y^2 \leq x + y + 2\,, \qquad x - 3y \leq 10\,.$$

$\left[\frac{125}{6}\right]$

Find the area of the region $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$x + y^2 - 2y + 2 \leq 0\,, \qquad y^2 + 2 \leq x + 4y\,.$$

$\left[\frac{1}{3}\right]$

Determine the integral $\displaystyle\iint_\Omega x \,\mathrm{d}x\,\mathrm{d}y$, where the region $\Omega \subset \mathbb{R}^2$ is given by the inequalities

$$x^2 + y^2 \leq 9\,, \qquad x^2 - y^2 \geq 1\,, \qquad x \geq 0\,.$$

$\left[\frac{32}{3}\right]$

Find the area of the region $\Omega \subset \mathbb{R}^2$ given by the inequality

$$2x^2 + 3y^2 \leq 4x\,.$$

$\left[\sqrt{\frac{2}{3}}\,\pi\right]$

Find the area of the region $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$\left(x^2 + y^2\right)^3 \leq 8xy(x^2 - y^2)\,, \qquad x\,, y \geq 0\,.$$

Use polar coordinates. $\left[\frac{1}{2}\right]$

Find the area of the region $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$\left(x + y\right)^4 \leq 4xy\,, \qquad x,\, y \geq 0\,.$$

Use the substitution $x = r\cos^2\varphi$, $y = r\sin^2\varphi$, where $r > 0$ and $0 < \varphi < \frac{1}{2}\pi$. $\left[\frac{1}{3}\right]$

Determine the integral $\displaystyle\iint_\Omega \frac{x}{y} \,\mathrm{d}x\,\mathrm{d}y$, where the region $\Omega \subset \mathbb{R}^2$ is given by the inequalities

$$1 \leq xy \leq 3\,, \qquad x \leq 2y \leq 8x\,.$$

Use the substitution $u = xy$ and $v = \dfrac{y}{x}$. $\left[\frac{7}{4}\right]$

Determine the coordinate x_{cg} of a centre of gravity of a homogenous area $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$x^2 - 8x - 2y + 12 \leq 0\,, \qquad x + y \leq 2\,,$$

provided you now that its area is equal to $\frac{2}{3}$. $\left[x_c = 3\right]$

Determine the coordinate y_{cg} of a centre of gravity of a homogenous area $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$4y^2 - y - 2 + 2x \leq 0\,, \qquad 2x + 3y + 6 \geq 0\,,$$

provided you now that its area is equal to 9. $\left[y_{cg} = \frac{1}{2}\right]$

Determine the coordinate y_{cg} of a centre of gravity of a homogenous area $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$y^2 + 2x + 4 \leq y\,, \qquad y^2 + 2y \leq x + 5\,,$$

provided you now that its area is equal to $\frac{27}{4}$. $\left[y_{cg} = -\frac{1}{2}\right]$

Determine the coordinate y_{cg} of a centre of gravity of a homogenous area $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$4x^2 + 9y^2 \leq 8x\,, \qquad y \geq 0\,,$$

provided you now that its area is equal to $\frac{1}{3}\,\pi$. $\left[y_{cg} = \frac{8}{9\pi}\right]$

Determine the coordinate x_{cg} of a centre of gravity of a homogenous area $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$1 \leq xy \leq 4\,, \qquad x \leq y \leq 9x\,,$$

provided you now that its area is equal to $3\ln 3$. Use the substitution $u = xy$ and $v = \dfrac{y}{x}$.
$\left[x_{cg} = \frac{28}{27\ln 3}\right]$

Determine the coordinate y_{cg} of a centre of gravity of a homogenous area $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$1 \leq xy \leq 4\,, \qquad x \leq y \leq 9x\,,$$

provided you now that its area is equal to $3\ln 3$. Use the substitution $u = xy$ and $v = \dfrac{y}{x}$
$\left[y_{cg} = \frac{28}{9\ln 3}\right]$

Determine the coordinate x_{cg} of a centre of gravity of a homogenous area $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$1 \leq xy^2 \leq 8\,, \qquad x \leq 27y \leq 27x\,,$$

provided you now that its area is equal to 9. Use the substitution $u = xy^2$ and $v = \dfrac{y}{x}$.
$\left[x_{cg} = \frac{182}{27}\right]$

Determine the coordinate y_{cg} of a centre of gravity of a homogenous area $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$1 \le xy^2 \le 8\,, \qquad x \le 27y \le 27x\,,$$

provided you now that its area is equal to 9. Use the substitution $u = xy^2$ and $v = \dfrac{y}{x}$.

$\left[y_{cg} = \frac{7}{9}\ln 3\right]$

Determine the coordinate y_{cg} of a centre of gravity of a homogenous area $\Omega \subset \mathbb{R}^2$ given by the inequalities

$$1 \le xy \le 4\,, \qquad x \le y^2 \le 2x\,,$$

provided you now that its area is equal to $\ln 2$. Use a substitution $u = xy$ and $v = \dfrac{y^2}{x}$.

$\left[y_{cg} = \frac{3}{4\ln 2}\left(4\sqrt[3]{4} - 1\right)\left(\sqrt[3]{2} - 1\right)\right]$

A force F spread over an area $\mathcal{S} \subset \mathbb{R}^2$ given by the inequalities

$$x^2 + y^2 \le 5\,, \qquad y \ge x^2 + 1\,,$$

causes a pressure $p(x, y) = y$. Find the total force F, i.e., the integral

$$F = \iint_{\mathcal{S}} p(x, y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

$\left[F = \frac{14}{5}\right]$

A force F spread over an area $\mathcal{S} \subset \mathbb{R}^2$ given by the inequalities

$$3x^2 + 2y^2 \le 6\,, \qquad x^2 + 2 \le 2y^2\,, \qquad x,\, y > 0\,,$$

causes a pressure $p(x, y) = 2y$. Find the total force F, i.e., the integral

$$F = \iint_{\mathcal{S}} p(x, y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

$\left[F = \frac{4}{3}\right]$

A force F spread over an area $\mathcal{S} \subset \mathbb{R}^2$ given by the inequalities

$$9x^2 + 4y^2 \le 36\,, \qquad y > 0\,,$$

causes a pressure $p(x, y) = y$. Find the total force F, i.e., the integral

$$F = \iint_{\mathcal{S}} p(x, y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

$\left[F = 12\right]$

Let $I = \displaystyle\int_0^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x$. Then the following equality holds:

$$I^2 = \int_0^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x \int_0^{\infty} \mathrm{e}^{-y^2}\,\mathrm{d}y = \iint_{\Omega} \mathrm{e}^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y\,,$$

where $\Omega \subset \mathbb{R}^2$ is given by the inequalities $x,\, y > 0$.
Using polar coordinates, find the integral I. $\left[I = \frac{1}{2}\sqrt{\pi}\right]$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + 4y^2 \le z \le 3\,.$$

$\left[\frac{9}{4}\pi\right]$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$0 \le z \le xy\,, \qquad x^2 + y^2 \le 4\,, \qquad x,\, y > 0\,.$$

$\left[2\right]$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 + z^2 \le 4\,, \qquad x^2 + y^2 - 2z^2 \ge 1\,.$$

$\left[4\pi\right]$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 \le z\,, \qquad 2x^2 + 2y^2 + z^2 \le 8\,.$$

$\left[\frac{2}{3}\pi\left(8\sqrt{2} - 7\right)\right]$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 \le 4\,, \qquad x + z \ge 0\,, \qquad x - z \ge 0\,.$$

$\left[\frac{32}{3}\right]$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 + z^2 \le 9\,, \qquad -1 \le z \le 2\,, \qquad x,\, y \ge 0\,.$$

$\left[6\pi\right]$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 + z^2 \le 5\,, \qquad x^2 + y^2 \ge 1\,.$$

$$\left[\frac{32}{3}\,\pi\right]$$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 \geq 1\,, \qquad x^2 + y^2 + 2 \leq 2z\,, \qquad x^2 + y^2 + z \leq 4\,.$$

$$\left[\frac{3}{4}\,\pi\right]$$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 + z^2 \leq 1\,, \qquad x^2 + y^2 \leq x\,.$$

$$\left[\frac{2}{9}\,(3\pi - 4)\right]$$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 \geq z^2\,, \qquad x^2 + y^2 \leq 2y \qquad z \geq 0\,.$$

$$\left[\frac{32}{9}\right]$$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$9x^2 + 4y^2 \leq z^2\,, \qquad 0 \leq z \leq 12 \qquad x \geq 0\,.$$

$$\left[4\pi\right]$$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 \leq 2x\,, \qquad y - z \geq 0\,, \qquad z \geq 0\,.$$

$$\left[\frac{2}{3}\right]$$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$4z^2 \leq x^2 + y^2 \leq 2x\,.$$

$$\left[\frac{32}{9}\right]$$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$4z + x^2 + y^2 \leq 24\,, \qquad x^2 + y^2 \leq 4z^2\,, \qquad x,\, z \geq 0\,.$$

$$\left[\frac{32}{3}\,\pi\right]$$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 \leq 2z\,, \qquad z - 4 \leq \sqrt{x^2 + y^2}\,.$$

$\left[\frac{128}{3}\,\pi\right]$

Find the volume of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$\left(x^2 + y^2 + z^2\right)^3 \le 12xyz\,, \qquad x,\, y,\, z > 0\,.$$

Use spherical coordinates. $\left[\frac{1}{2}\right]$

Find the coordinate z_{cg} of the centre of gravity of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + z^2 \le 1\,, \qquad y^2 + z^2 \le 1\,, \qquad z \ge 0\,,$$

provided you know that its volume is $V = \frac{8}{3}$. $\left[z_{cg} = \frac{3}{8}\right]$

Find the coordinate z_{cg} of the centre of gravity of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$0 \le z \le xy\,, \qquad x + y \le 1\,, \qquad x,\, y \ge 0\,,$$

provided you know that its volume is $V = \frac{1}{24}$. $\left[z_{cg} = \frac{1}{15}\right]$

Find the coordinate z_{cg} of the centre of gravity of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 + z^2 \le 9\,, \qquad -1 \le z \le 2\,,$$

provided you know that its volume is $V = 24\pi$. $\left[z_{cg} = \frac{13}{32}\right]$

Find the coordinate x_{cg} of the centre of gravity of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 \le 4\,, \qquad x^2 + z^2 \le 4\,, \qquad 0 \le x \le 1\,,$$

provided you know that its volume is $V = \frac{44}{3}$. $\left[x_{cg} = \frac{21}{44}\right]$

Find the coordinate x_{cg} of the centre of gravity of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + y^2 \le 2x\,, \qquad y - z \ge 0\,, \qquad z \ge 0\,,$$

provided you know that its volume is $V = \frac{2}{3}$. $\left[x_{cg} = 1\right]$

Find the coordinate z_{cg} of the centre of gravity of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$x^2 + 4y^2 \le z \le 3\,,$$

provided you know that its volume is $V = \frac{9}{4}\,\pi$. $\left[z_{cg} = 2\right]$

Find the coordinate x_{cg} of the centre of gravity of a solid $\mathcal{T} \subset \mathbb{R}^3$ given by the inequalities

$$9x^2 + 4y^2 \le z^2\,, \qquad 0 \le z \le 3\,, \qquad x \ge 0\,,$$

provided you know that its volume is $V = \frac{3}{4}\,\pi$. $\left[x_T = \frac{3}{\pi}\right]$

The charge density in a solid $\mathcal{T}$ given by the inequalities

$$x^2 + y^2 + z^2 \leq 2\,, \qquad x^2 + y^2 \leq 1\,, \qquad z \geq 0\,,$$

is $\rho(x, y, z) = z$. Find the total charge Q contained in $\mathcal{T}$, i.e., the integral

$$M = \iiint_{\mathcal{T}} \rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

$\left[M = \frac{3}{4}\,\pi\right]$

The charge density in a solid $\mathcal{T}$ given by the inequalities

$$x^2 + y^2 + 4 \leq 2z\,, \qquad x^2 + y^2 + z \leq 8\,, \qquad x^2 + y^2 \geq 1\,,$$

is $\rho(x, y, z) = \dfrac{1}{x^2 + y^2}$. Find the total charge Q contained in $\mathcal{T}$, i.e., the integral

$$Q = \iiint_{\mathcal{T}} \rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

$\left[Q = \frac{3}{2}\,\pi\big(8\ln 2 - 3\big)\right]$

The charge density in a solid $\mathcal{T}$ given by the inequalities

$$x^2 + y^2 \leq z^2\,, \qquad 0 \leq z \leq 1\,, \qquad x,\, y \geq 0\,,$$

is $\rho(x, y, z) = xy$. Find the total charge Q contained in $\mathcal{T}$, i.e., the integral

$$Q = \iiint_{\mathcal{T}} \rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

$\left[Q = \frac{1}{40}\right]$

The charge density in a solid $\mathcal{T}$ given by the inequalities

$$x^2 + y^2 + z^2 \leq 5\,, \qquad -1 \leq z \leq 2\,, \qquad x^2 + y^2 \geq 1\,,$$

is $\rho(x, y, z) = z$. Find the total charge Q contained in $\mathcal{T}$, i.e., the integral

$$Q = \iiint_{\mathcal{T}} \rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

$\left[Q = \frac{9}{4}\,\pi\right]$

Find the total electric charge Q of a ball $x^2 + y^2 + z^2 \leq 1$ with the charge density

$$\rho(x, y, z) = \mathrm{e}^{-\sqrt{x^2+y^2+z^2}}\,.$$

$$\left[Q = 4\pi\left(2 - 5\mathrm{e}^{-1}\right)\right]$$

A solid $\mathcal{T}$ given by the inequalities

$$x^2 + y^2 + z^2 \le 2z\,, \qquad x \ge 0\,,$$

has a density $\rho(x,y,z) = \sqrt{x^2 + y^2 + z^2}$. Find the total weight M of the solid $\mathcal{T}$, i.e., the integral

$$M = \iiint_{\mathcal{T}} \rho(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

$$\left[M = \tfrac{4}{5}\,\pi\right]$$

A solid $\mathcal{T} \subset \mathbb{R}^3$, is given by the inequalities

$$x^2 + y^2 \le 3z + 1\,, \qquad x^2 + y^2 + z \le 5\,,$$

and its density is $\rho(x,y,z) = 1$. Find its moment of inertia with respect to the axis z, i.e., the integral

$$J_z = \iiint_{\mathcal{T}} \rho(x,y,z)\left(x^2 + y^2\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

$$\left[J_z = \tfrac{128}{9}\,\pi\right]$$

A solid $\mathcal{T} \subset \mathbb{R}^3$, is given by the inequalities

$$x^2 + y^2 + z^2 \le 4\,, \qquad x^2 + y^2 - 2z^2 \ge 1\,, \qquad z \ge 0\,,$$

and its density is $\rho(x,y,z) = 1$. Find its moment of inertia with respect to the axis z, i.e., the integral

$$J_z = \iiint_{\mathcal{T}} \rho(x,y,z)\left(x^2 + y^2\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

$$\left[J_z = 2\pi\right]$$

A solid $\mathcal{T} \subset \mathbb{R}^3$, is given by the inequalities

$$x^2 + y^2 + z^2 \le 2\,, \qquad x^2 + y^2 \ge 1\,,$$

and its density is $\rho(x,y,z) = 1$. Find its moment of inertia with respect to the axis z, i.e., the integral

$$J_z = \iiint_{\mathcal{T}} \rho(x,y,z)\left(x^2 + y^2\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

$$\left[J_z = \tfrac{28}{15}\,\pi\right]$$

A solid $\mathcal{T} \subset \mathbb{R}^3$, is given by the inequalities

$$x^2 + y^2 \le 4\,, \qquad y + z \le 2\,, \qquad y - z \le 2$$

and its density is $\rho(x,y,z)=1$. Find its moment of inertia with respect to the axis z, i.e., the integral

$$J_z = \iiint_{\mathcal{T}} \rho(x,y,z)\big(x^2+y^2\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

$$\Big[J_z = 32\pi\Big]$$
