LECTURE 1

INDEFINITE INTEGRAL

1.1 Primitive function

Definition 1. Let f be a function defined on an open (bounded or unbounded) interval (a, b). Any function F such that

$$F'(x) = f(x) \qquad \text{for all } x \in (a, b),$$

is called **a primitive function to the function f on the interval** (a, b).

☛ ***Example 1.1.***

(a) $F(x) = \frac{1}{4}x^4$ is a primitive function to $f(x) = x^3$ on the interval $(-\infty, +\infty)$.

(b) $F(x) = -x^{-1}$ is a primitive function to $f(x) = x^{-2}$ on the interval $(-\infty, 0)$ and on the interval $(0, +\infty)$, but not e.g. on $(-1, 5)$ that contains $0 \notin D_f$.

Remark. If F is a primitive function to f on the interval (a, b), then also any function of the form $G(x) = F(x) + c$, where $c \in \mathbb{R}$ is any constant, is a primitive function to f on (a, b). The following theorem states that in this way, all primitive functions are exhausted:

Theorem. *If F, G are any two primitive functions to a function f on an interval (a, b), then there exists a constant $c \in \mathbb{R}$ such that*

$$G(x) = F(x) + c$$

for all $x \in (a, b)$.

Proof. Consider a function $H(x) = G(x) - F(x)$. Functions F, G are primitive functions to f, thus

$$H'(x) = (G(x) - F(x))' = G'(x) - F'(x) = f(x) - f(x) = 0.$$

According to the Cauchy Mean Theorem, $H(x)$ is constant on (a, b), i.e., $H(x) = c$.

Definition 2. The set of all primitive functions (if it is non-empty) to a function f on an interval (a, b) is called **an indefinite integral of the function f on the interval** (a, b) and it is denoted by the symbol $\int f$ or $\int f(x)\,\mathrm{d}x$.

Remark. If F is a primitive function to a function f on (a, b), we write

$$\int f(x)\,\mathrm{d}x = F(x) + c\,.$$

The constant c is called **a constant of integration**.

Theorem (Additivity of an integral with respect to the domain of integration).

1. *If a function f has an integral on an interval (a, b) and I is an open subinterval of (a, b), then f has an integral on I, too.*
2. *If a function f has an integral on intervals $I_1, I_2, \cdots, I_m$ and if their union $I = I_1 \cup I_2 \cup \cdots \cup I_m$ is an interval, then the function f has an integral on I.*

Theorem (Indefinite integral of a continuous function).
If a function $f(x)$ is continuous on an interval $\langle a, b\rangle$, then there exists its primitive function on (a, b).

Theorem (Indefinite integral of a derivative).
If $f'(x)$ is continuous on an interval (a, b), it is

$$\int f'(x)\,\mathrm{d}x = f(x) + c.$$

Theorem (Linearity of an integral)

1. *Let F, G be primitive functions on an interval (a,b) to f, g, let r be a number. Then $F+G$ is a primitive function to $f+g$ and rF is a primitive function to rf on (a,b).*
2. *Let f, g have indefinite integrals on an interval (a,b), let r be a number. Then $f+g$ and rf have indefinite integrals on (a,b), too, and the following equalities are satisfied:*

$$\begin{aligned}\int (f(x)+g(x))\,\mathrm{d}x &= \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x\,,\\ \int rf(x)\,\mathrm{d}x &= r\int f(x)\,\mathrm{d}x\,.\end{aligned}$$

3. *Let functions $f_1, f_2, \cdots, f_m$ have indefinite integrals on (a,b), let $r_1, r_2, \cdots, r_m$ be any constants. Then the function $r_1f_1 + r_2f_2 + \cdots + r_mf_m$ has an indefinite integral, too, and the following equality is satisfied:*

$$\int (r_1f_1(x)+r_2f_2(x)+\cdots+r_nf_n(x))\,\mathrm{d}x =$$

$$= r_1\int f_1(x)\,\mathrm{d}x + r_2\int f_2(x)\,\mathrm{d}x + \cdots + r_n\int f_n(x)\,\mathrm{d}x.$$

1.2 Fundamental integration formulas

The well-known formulas from the differential calculus imply:

$$1)\quad \int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \qquad x \in \mathbb{R} \text{ for } n \in \mathbb{Z},\ n > 0;$$

$$x \in \mathbb{R} \setminus \{0\},\ n \in \mathbb{Z},\ n < -1,$$

$$x > 0 \text{ for } n \in \mathbb{R},\ n \notin \mathbb{Z}.$$

$$2)\quad \int \frac{dx}{x} = \ln|x| + c, \qquad x \in \mathbb{R} \setminus \{0\}.$$

$$3)\quad \int e^x \, dx = e^x + c; \qquad x \in \mathbb{R}.$$

$$4)\quad \int a^x \, dx = \frac{a^x}{\ln a} + c, \qquad x \in \mathbb{R},\ a > 0,\ a \neq 1.$$

$$5)\quad \int \sin x \, dx = -\cos x + c, \qquad x \in \mathbb{R}.$$

$$6)\quad \int \cos x \, dx = \sin x + c, \qquad x \in \mathbb{R}.$$

7) $$\int \frac{1}{\cos^2 x}\,\mathrm{d}x = \tan x + c,$$

$$x \in \left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right), k \in \mathbb{Z}.$$

8) $$\int \frac{1}{\sin^2 x}\,\mathrm{d}x = -\cot x + c,$$

$$x \in (2k\pi, (2k+1)\pi),\ k \in \mathbb{Z}.$$

9) $$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \begin{cases} \arcsin x + c, \\ -\arccos x + c, \end{cases} \qquad x \in (-1, 1).$$

10) $$\int \frac{1}{1+x^2}\,\mathrm{d}x = \begin{cases} \arctan x + c, \\ -\mathrm{arccot}\ x + c, \end{cases} \qquad x \in \mathbb{R}.$$

11) $$\int \cosh x\,\mathrm{d}x = \sinh x + c; \qquad x \in \mathbb{R}.$$

12) $$\int \sinh x\,\mathrm{d}x = \cosh x + c; \qquad x \in \mathbb{R}.$$

13) $$\int \frac{1}{\cosh^2 x}\,\mathrm{d}x = \tanh x + c,$$

14) $$\int \frac{1}{\sinh^2 x}\,\mathrm{d}x = \coth x + c,$$

15) $$\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x = \operatorname{argsinh} x + c = \ln\left(x + \sqrt{x^2+1}\right) + c$$

16) $$\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x = \operatorname{argcosh} x + c = \ln\left(x + \sqrt{x^2-1}\right) + c$$

17) $$\int \frac{1}{1-x^2}\,\mathrm{d}x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|,$$

$$\ln\sqrt{\frac{1+x}{1-x}} = \operatorname{argtanh} x, \quad \ln\sqrt{\frac{x+1}{x-1}} = \operatorname{argcoth} x$$

☛ ***Example 1.2.***

Find the following integrals:

(a) $I = \int (5x^4 - 2x^3 - 3x + 7)\,\mathrm{d}x$

Solution. Due to the linearity of an integral, we can write:

$$I = \int (5x^4 - 2x^3 - 3x + 7)\,\mathrm{d}x =$$

$$= 5\int x^4\,\mathrm{d}x - 2\int x^3\,\mathrm{d}x - 3\int x\,\mathrm{d}x + 7\int 1\,\mathrm{d}x =$$

$$= 5\left(\frac{x^5}{5} + c_1\right) - 2\left(\frac{x^4}{4} + c_2\right) - 3\left(\frac{x^2}{2} + c_4\right) + 7(x + c_5) =$$

$$= x^5 - \frac{1}{2}x^4 - \frac{3}{2}x^2 + 7x + c\,, \qquad x \in \mathbb{R}\,.$$

(b) $I = \int \frac{2x^3 - 3\sqrt{x} + 5}{x}\,\mathrm{d}x$

Solution. We can divide the numerator by the denominator and then use the basic formulas:

$$I = \int \frac{2x^3 - 3\sqrt{x} + 5}{x}\,\mathrm{d}x = \int (2x^2 - 3x^{-\frac{1}{2}} + 5x^{-1})\,\mathrm{d}x =$$

$$= 2\int x^2\,\mathrm{d}x - 3\int x^{-\frac{1}{2}}\,\mathrm{d}x + 5\int x^{-1}\,\mathrm{d}x =$$

$$= 2\left(\frac{x^3}{3} + c_1\right) - 3\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c_2\right) + 5\left(\ln|x| + c_3\right) =$$

$$= \frac{2}{3}x^3 - 6\sqrt{x} + \ln|x| + c\,, \qquad x \in (0, +\infty)\,.$$

(c) $I = \int \mathrm{e}^{4x}\,\mathrm{d}x$

Solution. Notice that $(\mathrm{e}^{4x})' = 4\mathrm{e}^{4x}$. To obtain the given function after the differentiation of the result, it must be

$$I = \int \mathrm{e}^{4x}\,\mathrm{d}x = \frac{\mathrm{e}^{4x}}{4} + c\,, \qquad x \in \mathbb{R}\,.$$

(d) $I = \int \cos(3x-2)\,\mathrm{d}x$

Solution. Since $(\cos(3x-2))' = 3\sin(3x-2)$, it is:

$$I = \int \cos(3x-2)\,\mathrm{d}x = \frac{\sin(3x-2)}{3} + c\,, \qquad x \in \mathbb{R}\,.$$

1.3 Integration per partes (by parts)

Theorem (Itegration per partes). *Let functions u, v have continuous derivatives on an interval (a, b). Then*

$$\int u'(x)v(x)\,\mathrm{d}x = u(x)v(x) - \int u(x)v'(x)\,\mathrm{d}x\,. \tag{1.1}$$

Proof. The formula for the derivative of a forduct implies:

$$(uv)' = u'v + uv' \qquad \text{on} \quad (a, b),$$

thus

$$\begin{aligned} \int (uv)' &= \int (u'v + uv') \\ uv + c &= \int u'v + \int uv' \\ \int u'v &= uv - \int uv' + c \end{aligned}$$

The last equality is equivalent with the stated formula.

Application I – simplification of the given function

☛ ***Example 1.3.***

Find the integral $\int x \cos x \, \mathrm{d}x$.

Solution. To simplify the function that should be integrated, let us choose x for the differentiation:

$$\int x \cos x \, \mathrm{d}x = \left| \begin{array}{ll} u' = \cos x\,, & u = \sin x \\ v = x\,, & v' = 1 \end{array} \right| =$$

$$= x \sin x - \int \sin x \, \mathrm{d}x = x \sin x + \cos x + c\,, \qquad x \in \mathbb{R}\,.$$

☛ ***Example 1.4.***

Find the integral $\int x^2 \sin x \, \mathrm{d}x$.

Solution. Bt a repeated differentiation of x^2 we simplify the integrand to a single goniometric function:

$$\int x^2 \sin x \, \mathrm{d}x = \left| \begin{array}{ll} u' = \sin x \,, & u = -\cos x \\ v = x^2 \,, & v' = 2x \end{array} \right| =$$

$$= x^2(-\cos x) - \int 2x(-\cos x) \, \mathrm{d}x = -x^2 \cos x + 2\int x \cos x \, \mathrm{d}x =$$

$$= \left| \begin{array}{ll} u' = \cos x \,, & u = \sin x \\ v = x \,, & v' = 1 \end{array} \right| = -x^2 \cos x + 2 \left(x \sin x - \int \sin x \, \mathrm{d}x \right) =$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c \,, \qquad x \in \mathbb{R} \,.$$

☛ ***Example 1.5.***

Find the integral $\int (x^2 + 2x - 5)\, e^{3x}\, dx$.

Solution.

$$\int (x^2 + 2x - 5)\, e^{3x}\, dx = \left| \begin{array}{ll} u' = e^{3x}, & u = \frac{1}{3} e^{3x} \\ v = x^2 + 2x - 5, & v' = 2x + 2 \end{array} \right| =$$

$$= \frac{1}{3}\left(x^2 + 2x - 5\right) e^{3x} - \frac{1}{3} \int (2x + 2) e^{3x}\, dx =$$

$$= \left| \begin{array}{ll} u' = e^{3x}, & u = \frac{1}{3} e^{3x} \\ v = 2x + 2, & v' = 2 \end{array} \right| =$$

$$= \frac{1}{3}\left(x^2 + 2x - 5\right) e^{3x} - \frac{1}{3}\left(\frac{1}{3}(2x + 2) e^{3x} - \frac{2}{3} \int e^{3x}\, dx\right) =$$

$$= \frac{1}{3}\left(x^2 + 2x - 5\right) e^{3x} - \frac{1}{3}\left(\frac{1}{3}(2x + 2) e^{3x} - \frac{2}{9} e^{3x}\right) + c =$$

$$= \left(\frac{1}{3} x^2 + \frac{4}{9} x - \frac{49}{27}\right) e^{3x} + c, \qquad x \in \mathbb{R}.$$

☛ ***Example 1.6.***

Find the integral $\int x^3 \mathrm{e}^{5x}\,\mathrm{d}x$.

Solution.

$$\int x^3 \mathrm{e}^{5x}\,\mathrm{d}x = \left| \begin{array}{ll} u' = \mathrm{e}^{5x}, & u = \frac{1}{5}\mathrm{e}^{5x} \\ v = x^3, & v' = 3x^2 \end{array} \right| =$$

$$= \frac{1}{5}x^3\mathrm{e}^{5x} - \frac{3}{5}\int x^2\mathrm{e}^{5x}\,\mathrm{d}x = \left| \begin{array}{ll} u' = \mathrm{e}^{5x}, & u = \frac{1}{5}\mathrm{e}^{5x} \\ v = x^2, & v' = 2x \end{array} \right| =$$

$$= \frac{1}{5}x^3\mathrm{e}^{5x} - \frac{3}{5}\left(\frac{1}{5}x^2\mathrm{e}^{5x} - \frac{2}{5}\int x\mathrm{e}^{5x}\,\mathrm{d}x\right) = \left| \begin{array}{ll} u' = \mathrm{e}^{5x}, & u = \frac{1}{5}\mathrm{e}^{5x} \\ v = x, & v' = 1 \end{array} \right| =$$

$$= \frac{1}{5}x^3\mathrm{e}^{5x} - \frac{3}{25}x^2\mathrm{e}^{5x} + \frac{6}{25}\left(\frac{1}{5}x\mathrm{e}^{5x} - \frac{1}{5}\int \mathrm{e}^{5x}\,\mathrm{d}x\right) =$$

$$= \frac{1}{5}x^3\mathrm{e}^{5x} - \frac{3}{25}x^2\mathrm{e}^{5x} + \frac{6}{125}x\mathrm{e}^{5x} - \frac{6}{625}\mathrm{e}^{5x} + c, \qquad x \in \mathbb{R}.$$

Remark. In the previous examples, we have calculated:

$$\int (x^2 + 2x - 5)\,\mathrm{e}^{3x}\,\mathrm{d}x = \left(\tfrac{1}{3}x^2 + \tfrac{4}{9}x - \tfrac{49}{27}\right)\mathrm{e}^{3x} + c\,,$$

$$\int x^3\mathrm{e}^{5x}\,\mathrm{d}x = \left(\tfrac{1}{5}x^3 - \tfrac{3}{25}x^2 + \tfrac{6}{125}x - \tfrac{6}{625}\right)\mathrm{e}^{5x} + c\,.$$

In general:

$$\int P(x)\mathrm{e}^{kx}\,\mathrm{d}x = Q(x)e^{kx}\,.$$

Similar exercises can therefore be solved without a repeated integration by parts simply by finding a polynomial $Q(x)$ such that

$$\left(Q(x)e^{kx}\right)' = P(x)\mathrm{e}^{kx}.$$

☛ ***Example 1.7.***

Using the estimation method, find the integral $\int (x^2 + 2x - 5)\, e^{3x}\, dx$

Solution.

$$\int (x^2 + 2x - 5)\, e^{3x}\, dx = (Ax^2 + Bx + C)\, e^{3x}\,.$$

It is sufficient to find constants $A,\ B,\ C$ such that

$$\Big(\big(Ax^2 + Bx + C\big)\, e^{3x}\Big)' = \big(x^2 + 2x - 5\big)\, e^{3x}\,,$$

thus

$$\begin{aligned}
(2Ax + B)\, e^{3x} + \big(Ax^2 + Bx + C\big) \cdot 3e^{3x} &= (x^2 + 2x - 5)\, e^{3x} \\
\big(3Ax^2 + (2A + 3B)x + (B + 3C)\big)\, e^{3x} &= (x^2 + 2x - 5)\, e^{3x} \\
3Ax^2 + (2A + 3B)x + (B + 3C) &= x^2 + 2x - 5
\end{aligned}$$

$$3Ax^2 + (2A + 3B)x + (B + 3C) = x^2 + 2x - 5$$

To obtain equal functions on both sides, the coefficients by the same powers of x must be equal:

$$\begin{array}{lcrcl} x^2 & \ldots & 3A & = & 1 \\ x^1 & \ldots & 2A + 3B & = & 2 \\ x^0 & \ldots & B + 3C & = & -5 \end{array}$$

The solution of this system of linear equations is

$$A = \tfrac{1}{3}, \quad B = \tfrac{4}{9}, \quad C = -\tfrac{49}{27},$$

thus

$$\int (x^2 + 2x - 5)\,\mathrm{e}^{3x}\,\mathrm{d}x = \left(\tfrac{1}{3}x^2 + \tfrac{4}{9}x - \tfrac{49}{27}\right)\mathrm{e}^{3x} + c\,.$$

☛ ***Example 1.8.***

Find the integral $\int x^5 \ln x \, \mathrm{d}x$.

Solution. Notice that the given function will be simplified by the differentiation of $\ln x$:

$$\int x^5 \ln x \, \mathrm{d}x = \left| \begin{array}{ll} u' = x^5 \,, & u = \dfrac{x^6}{6} \\ v = \ln x \,, & v' = \dfrac{1}{x} \end{array} \right| = \tfrac{1}{6} x^6 \ln x - \tfrac{1}{6} \int \frac{x^6}{x} \, \mathrm{d}x =$$

$$= \tfrac{1}{6} x^6 \ln x - \tfrac{1}{6} \int x^5 \, \mathrm{d}x = \tfrac{1}{6} x^6 \ln x - \tfrac{1}{36} x^6 + c \,, \qquad x \in \mathbb{R} \,.$$

☛ ***Example 1.9.***

Find the integral $\int \ln x \, \mathrm{d}x$.

Solution. The given function does not look like a product. Nevertheless, we can again use the integration by parts. It would we helpful to replace $\ln x$ by its derivative $1/x$. To achieve this, it is sufficient to consider the integrand as a product $1 \ln x$:

$$\int 1 \ln x \, \mathrm{d}x = \left| \begin{array}{ll} u' = 1 \, , & u = x \\ v = \ln x \, , & v' = \dfrac{1}{x} \end{array} \right| =$$

$$= x \ln x - \int \frac{x}{x} \, \mathrm{d}x = x \ln x - \int 1 \, \mathrm{d}x = x \ln x - x + c \, , \qquad x \in \mathbb{R} \, .$$

Indirect determination of an indefinite integral: per partes leading to the solution of an equation

In some cases we can avoid a direct integration by the repeated use of per partes method and solving a simple equation for the integral (typical cases cover $\mathrm{e}^x, \sin x, \cos x$.:

$$I = h(x) + kI\,.$$

☛ ***Example 1.10.***

Find the integral $\int \mathrm{e}^x \cos x \, \mathrm{d}x$.

Solution.

$$\int \mathrm{e}^x \cos x \, \mathrm{d}x = \left| \begin{array}{ll} u' = \mathrm{e}^x \,, & u = \mathrm{e}^x \\ v = \cos x \,, & v' = -\sin x \end{array} \right| =$$

$$= \mathrm{e}^x \cos x + \int \mathrm{e}^x \sin x \, \mathrm{d}x = \left| \begin{array}{ll} u' = \mathrm{e}^x \,, & u = \mathrm{e}^x \\ v = \sin x \,, & v' = \cos x \end{array} \right| =$$

$$= \mathrm{e}^x \cos x + \left(\mathrm{e}^x \sin x - \int \mathrm{e}^x \cos x \, \mathrm{d}x\right)$$

Thus we have obtained an equation

$$\begin{aligned} \int \mathrm{e}^x \cos x \, \mathrm{d}x &= \mathrm{e}^x \cos x + \mathrm{e}^x \sin x - \int \mathrm{e}^x \cos x \, \mathrm{d}x \\ 2 \int \mathrm{e}^x \cos x \, \mathrm{d}x &= \mathrm{e}^x \cos x + \mathrm{e}^x \sin x \\ \int \mathrm{e}^x \cos x \, \mathrm{d}x &= \frac{1}{2} (\mathrm{e}^x \cos x + \mathrm{e}^x \sin x) \end{aligned}$$

Remark. Integrals of the type $\int e^{kx}(P(x)\cos\omega x + Q(x)\sin\omega x)\,dx$ where $P(x)$ are $Q(x)$ polynomials of degree n_1 and n_2, respectively, and k and ω are not both equal to zero, are always equal to

$$\int e^{kx}(P(x)\cos\omega x + Q(x)\sin\omega x)\,dx = e^{kx}(R(x)\cos\omega x + S(x)\sin\omega x),$$

where $R(x)$ and $S(x)$ are polynomials of degree $n = \max(n_1; n_2)$ with unknown coefficients that can be found with the use of a derivative – see p. 19.

☛ ***Example 1.11.***

Find the integral $\int \mathrm{e}^{-x}(3\cos 2x - (4x+1)\sin 2x)\,\mathrm{d}x$.

Solution.

We are looking for the solution in the form

$$\mathrm{e}^{-x}\left((Ax+B)\cos 2x + (Cx+D)\sin 2x\right) .$$

Using a derivative and comparing the coefficients by particular powers of x we obtain

$$\int \mathrm{e}^{-x}(3\cos 2x - (4x+1)\sin 2x)\,\mathrm{d}x =$$

$$= \mathrm{e}^{-x}\left(\left(\tfrac{8}{5}x + \tfrac{11}{25}\right)\cos 2x + \left(\tfrac{4}{5}x + \tfrac{23}{25}\right)\sin 2x\right) .$$

Remark.

Integrals of the type

$$\int \sin ax \ \cos bx \, \mathrm{d}x, \quad \int \sin ax \ \sin bx \, \mathrm{d}x \ ,$$

$$\int \cos ax \ \cos bx \, \mathrm{d}x, \quad a \neq b,$$

can be solved using a multiple use of the per partes method, or we can simplify the function using the formulas

$$\sin\alpha \ \cos\beta = (\sin(\alpha+\beta) + \sin(\alpha-\beta))/2,$$
$$\sin\alpha \ \sin\beta = (\cos(\alpha-\beta) - \cos(\alpha+\beta))/2,$$
$$\cos\alpha \ \cos\beta = (\cos(\alpha+\beta) + \cos(\alpha-\beta))/2.$$

☛ ***Example***

$$\int \sin 5x \cos x \, \mathrm{d}x = (1/2) \int (\sin 6x + \sin 4x) \, \mathrm{d}x =$$

$$= -(1/12) \cos 6x - (1/8) \cos 4x + c.$$

1.4 Substitution in the indefinite integral

Theorem (The first theorem on the substitution) *Let the integral on the left side of the equation*

$$\int f(x)\,\mathrm{d}x = \int f(\varphi(t))\varphi'(t)\,\mathrm{d}t \tag{1.2}$$

exists on an interval J *and is equal to* $F(x)$*. Let* $x = \varphi(t)$ *have a continuous derivative on an interval* $\varphi(I) \subset J$*. Then the integral on the right side of the equation (1.2) exists on* I *and is equal to* $F\big(\varphi(t)\big)$*.*

Proof. Let F be a primitive function to a function f on an interval J. Since φ maps the interval I to the interval J, the composite functions $F(\varphi(t))$ and $f(\varphi(t))$ are defined on I and the rule for the derivative of a composite function implies

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\varphi(t)) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t)\,, \quad t \in I.$$

Remark. The previous theorem is useful in the cases where the integral is "prepared"in the form

$$\int f(\varphi(t))\varphi'(t)\,\mathrm{d}t\,.$$

☛ ***Example 1.12.***

Find the integral $\int \mathrm{e}^{5t+3}\,\mathrm{d}t\,.$

Solution. The integrand is continuous in $\mathbb{R}$, the integral therefore existuje. Denote

$$x = \varphi(t) = 5t + 3, \qquad \mathrm{d}x = \varphi'(t)\,\mathrm{d}t = 5\,\mathrm{d}t\,.$$

All assumptions of the theorem on the substitution are satisfied and we can write:

$$\int \mathrm{e}^{5t+3}\,\mathrm{d}t = \frac{1}{5}\int \mathrm{e}^{5t+3}5\,\mathrm{d}t = \frac{1}{5}\int \mathrm{e}^{x}\,\mathrm{d}x = \frac{1}{5}\mathrm{e}^{x} + c =$$

$$= \frac{1}{5}\mathrm{e}^{5t+3} + c\,, \quad t \in \mathbb{R}\,.$$

☛ ***Example 1.13.***

Find the integral $\displaystyle\int \frac{e^t}{(e^t+2)^3}\,dt$.

Solution. The given function is continuous in $\mathbb{R}$, the integral therefore exists in $\mathbb{R}$,. Similarly as in the previous example,

$$\int \frac{e^t}{(e^t+2)^3}\,dt = \left| \begin{array}{c} x = e^t + 2 \\ dx = e^t\,dt \end{array} \right| = \int \frac{1}{x^3}\,dx = \int x^{-3}\,dx =$$

$$= \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c = -\frac{1}{2(e^t+2)^2} + c\,, \quad t \in \mathbb{R}\,.$$

Remark. Of course, it does not matter which letters are used for the variables.

☛ ***Example 1.14.***

Find the integral $\displaystyle\int \frac{(\ln x)}{x\cdot\sqrt{(5+\ln^2 x)^3}}\,\mathrm{d}x$ on $I=(0,+\infty)$.

Solution.

$$\int \frac{(\ln x)}{x\cdot\sqrt{(5+\ln^2 x)^3}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(5+\ln^2 x)^3}}\,(\ln x)\,\frac{1}{x}\,\mathrm{d}x =$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{(5+\ln^2 x)^3}}\,(2\ln x)\,\frac{1}{x}\,\mathrm{d}x = \left|\begin{array}{l} t = 5+\ln^2 x \\ \mathrm{d}t = (2\ln x)\,\frac{1}{x}\,\mathrm{d}x \end{array}\right| =$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{t^3}}\,\mathrm{d}t = \frac{1}{2}\int t^{-\frac{3}{2}}\,\mathrm{d}t = \frac{t^{-\frac{1}{2}}}{-\frac{1}{2}} + c = -2\sqrt{t} + c =$$

$$= -2\sqrt{5+\ln^2 x} + c\,, \quad x \in (0,+\infty)\,.$$

Theorem (The second theorem on the substitution) *Let the integral on the left side of the equation*

$$\int f(\varphi(t))\varphi'(t)\,\mathrm{d}t = \int f(x)\,\mathrm{d}x \tag{1.3}$$

exist on an interval I and is equal to $F(t)$, let a function $x = \varphi(t)$ be such that it has a non-zero derivative at each point of I a maps I to $J = \varphi(I)$. Then the integral on the right side of the equation (1.3) exists on J and is equal to $F\big(\psi(x)\big)$, where $\psi(x)$ is an inverse function to the function $x = \varphi(t)$.

Proof. The function φ is invertible on I. Denote its inverse function by $t = \psi(x)$. This function maps an interval J on an interval I. According to the assumption, there exists a function G such that $G'(t) = f(\varphi(t))\varphi'(t)$, $t \in I$. Denote

$$F(x) = G(\psi(x)) .$$

The rule for the derivative of a composite function implies that $F'(x) = G'(\psi(x))\psi'(x) = G'(t)\psi'(x) = f(\varphi(t))\varphi'(t)\cdot 1/\varphi'(t) = f(x)$, $x \in J$.

Remark. Instead of $\varphi'(t) \neq 0$ for all $t \in I$ it is sufficient to require that φ is strictly monotonous and $\varphi'(t) = 0$ for at most a finite number of values of $t \in I$.

☛ ***Example 1.15.***

Find the integral $\int \dfrac{5}{\sqrt{1-x^2}}\,\mathrm{d}x\,,\, x \in (-1,1)$.

Solution. The integrand is continuous on the interval $J = (-1,1)$, the integral therefore exists takže integrál on J. To remove the square root, we can use the identity $\cos^2 t = 1 - \sin^2 t$ and the substitution:

$$\begin{aligned}
&x = \varphi(t) = \sin t\,; \quad \varphi(t) \text{ maps } (-\pi/2, \pi/2) \text{ on } (-1,1)\,;\\
&\mathrm{d}x = \cos t\,\mathrm{d}t\,; \quad \varphi'(t) = (\sin t)' = \cos t \neq 0\,;\\
&\sqrt{1-x^2} = |\cos t| = \cos t > 0\,;\\
&t = \arcsin x
\end{aligned}$$

$$\int \frac{5}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{5}{\cos t}\cos t\,\mathrm{d}t = 5\int 1\,\mathrm{d}t = 5t + c =$$
$$= 5\arcsin x + t, \quad x \in (-1,1)\,.$$