# LECTURE 2

# FURTHER METHODS OF INTEGRATION

# Integration of rational functions

**Theorem (Decomposition to partial fractions)**

*Let $P(x)$, $Q(x)$ be polynomials such that*

$$Q(x) = a(x-x_1)^{k_1} \cdots (x-x_r)^{k_r} \cdot (x^2+2p_1x+q_1)^{l_1} \cdots (x^2+2p_sx+q_s)^{l_s}$$

*where polynomials $x^2 + 2p_ix + q_i$, $i = 1, \ldots, s$, are irreducible in the field of real numbers. Then $R(x)$ can uniquely (up to the order of summands) be expressed in the form*

$$R(x) = \frac{P(x)}{Q(x)} = p(x) + \sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{A_{ij}}{(x-x_i)^j} + \sum_{i=1}^{s} \sum_{j=1}^{l_k} \frac{B_{ij}x + C_{ij}}{(x^2+2p_ix+q_i)^j},$$

*where $p(x)$ is a polynomial, $A_{ij}, B_{ij}, C_{ij}$ are real constants; this equality holds for all $x \in \mathsf{D}_R$, i.e., for all real $x$ different from the roots of the polynomial $Q(x)$.*

☛ ***Example 1.1.***

Decompose the function

$$R(x) = \frac{3x+5}{x^2-3x+2}$$

to the sum of partial fractions.

***Solution.*** The roots of the polynomial in the denominator are $1$ and $2$, thus

$$R(x) = \frac{3x+5}{x^2-3x+2} = \frac{3x+5}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}\,.$$

To find the coefficients $A, B$, bring the fractions on the right side to the common denominator. The function in the numerator must be identical with the given function $3x+5$, i.e., we require:

$$\frac{3x+5}{(x-1)(x-2)} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$3x+5 = (A+B)x + (-2A-B)$$

$$x^1 \quad \dots \quad 3 = A+B$$

$$x^0 \quad \dots \quad 5 = -2A-B$$

$$A = -8, \quad B = 11$$

Thus

$$R(x) = \frac{3x+5}{x^2-3x+2} = -\frac{8}{x-1} + \frac{11}{x-2}\,.$$

☛ ***Example 1.2.***

Decompose the following function to the sum partial fractions:

$$R(x) = \frac{3x+5}{(x-1)^2(x-2)}$$

***Solution.***

$$R(x) = \frac{3x+5}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

$$\frac{3x+5}{(x-1)^2(x-2)} = \frac{A(x-1)(x-2) + B(x-2) + C(x-1)^2}{(x-1)(x-2)}$$

$$3x+5 = A(x^2-3x+2) + B(x-2) + C(x^2-2x+1)$$

$$3x+5 = (A+C)x^2 + (-3A+B-2C)x + (2A-2B+C)$$

The functions on both sides must be identical, i.e., it must be:

$$\begin{array}{rcl} x^2 \;\ldots\; 0 & = & A + C \qquad\qquad (\rightarrow\; A = -C) \\ x^1 \;\ldots\; 3 & = & -3A + B - 2C \\ x^0 \;\ldots\; 5 & = & 2A - 2B + C \end{array}$$

$$\begin{array}{rcl} 3 & = & -A + B \\ 5 & = & A - 2B \end{array}$$

$$B = -8, \quad A = -11, \quad C = 11$$

Thus

$$R(x) = \frac{3x+5}{(x-1)^2(x-2)} = -\frac{11}{x-1} - \frac{8}{(x-1)^2} + \frac{11}{x-2}\,.$$

***Remark.*** Notice that if we try to decompose the function in the form

$$R(x) = \frac{3x+5}{(x-1)^2(x-2)} = \frac{A}{(x-1)^2} + \frac{B}{x-2},$$

we would obtain the condition

$$\begin{aligned} 3x+5 &= A(x-2) + B(x-1)^2 \\ 3x+5 &= A(x-2) + B(x^2-2x+1) \\ 3x+5 &= Bx^2 + (A-2B)x + (-2A+B) \end{aligned}$$

and the corresponding system of equations that has no solution (we have only two variables and three linearly independent equations):

$$\begin{array}{lrcl} x^2 \ldots & 0 &=& B \\ x^1 \ldots & 3 &=& A-2B \\ x^0 \ldots & 5 &=& -2A+B \\ \hline \end{array}$$

$$B = 0, \quad A = 3 \,\wedge\, A = -\frac{5}{2}$$

# The calculation of the integral of rational functions

The given rational function can first be decomposed to the sum of partial fractions. If the degree of the polynomial in the denominator is not higher than the degree of the polynomial in the numerator, it is necessary to start with the partial division (with the rest) of the polynomials.

The individual partial fractions can easily be integrated:

$$\int \frac{1}{x-a}\,\mathrm{d}x = \ln|x-a| + c$$

for $n \in \mathbb{N},\ n > 1,\ x \in (-\infty, a)$ or $x \in (a, \infty)$ it is:

$$\int \frac{1}{(x-a)^n}\,\mathrm{d}x = \frac{1}{1-n}\frac{1}{(x-a)^{n-1}} + c$$

☛ ***Example 1.3.***

Find the integral $\int \frac{\mathrm{d}x}{2-3x}$.

***Solution.***

$$\int \frac{\mathrm{d}x}{2-3x} = -\frac{1}{3}\int \frac{1}{2-3x}(-3)\,\mathrm{d}x = -\frac{1}{3}\ln|2-3x| + c.$$

☛ ***Example 1.4.***

Find the integral $\int \frac{\mathrm{d}x}{(2x+5)^3}$.

***Solution.***

$$\int \frac{\mathrm{d}x}{(2x+5)^3} = \frac{1}{2}\int \frac{1}{(2x+5)^3}\,2\,\mathrm{d}x = \frac{1}{2}\int (2x+5)^{-3}\,2\,\mathrm{d}x =$$

$$= \frac{1}{2}\cdot\frac{(2x+5)^{-2}}{-2} + c = -\frac{1}{4(2x+5)^2} + c\,.$$

☛ ***Example 1.5.***

Find the integral $I = \int \frac{3x+5}{(x-1)^2(x-2)}\,\mathrm{d}x$ .

***Solution.*** After the decomposition to partial fractions we obtain:

$$I = \int \left( -\frac{11}{x-1} - \frac{8}{(x-1)^2} + \frac{11}{x-2} \right) \mathrm{d}x =$$

$$= -11 \ln|x-1| - 8\frac{(x-1)^{-1}}{-1} + 11 \ln|x-2| + c =$$

$$= 11 \ln\left|\frac{x-2}{x-1}\right| + \frac{8}{x-1} + c \, .$$

The following examples illustrate what to do when the denominator contains a quadratic polynomial that has no real roots. If the numerator contains only a constant, we head towards the function $\arctan$ .

☛ ***Example 1.6.***

Find the integral $I = \int \frac{dx}{2+x^2}$.

***Solution.*** The polynomial in the denominator cannot be decomposed in the field of real numbers. The integration leads to $\arctan$ :

$$I = \int \frac{dx}{x^2+2} = \int \frac{dx}{2\left(\frac{x^2}{2}+1\right)} =$$

$$= \frac{\sqrt{2}}{2} \int \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1} \frac{1}{\sqrt{2}}\, dx = \frac{\sqrt{2}}{2} \arctan \frac{x}{\sqrt{2}} + c\,.$$

Similarly, we obtain

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan \frac{x}{a} + c, \quad a > 0, \quad x \in \mathbb{R}$$

☛ ***Example 1.7.***

Find the integral $I = \displaystyle\int \frac{\mathrm{d}x}{3x^2+2}$.

***Solution.***

$$I = \int \frac{\mathrm{d}x}{3x^2+2} = \int \frac{\mathrm{d}x}{2\left(\frac{3x^2}{2}+1\right)} =$$

$$= \frac{\sqrt{2}}{2\sqrt{3}} \int \frac{1}{\left(x\sqrt{\frac{3}{2}}\right)^2+1} \sqrt{\frac{3}{2}}\,\mathrm{d}x = \frac{1}{\sqrt{6}} \arctan\left(x\sqrt{\frac{3}{2}}\right) + c\,.$$

☛ ***Example 1.8.***

Find the integral $I = \int \dfrac{\mathrm{d}x}{2x^2 - 3x + 6}$.

***Solution.***

$$I = \int \frac{\mathrm{d}x}{2x^2 - 3x + 6} = \frac{1}{2}\int \frac{\mathrm{d}x}{x^2 - \frac{3}{2}x + 3} =$$

$$= \frac{1}{2}\int \frac{\mathrm{d}x}{\left(x - \frac{3}{4}\right)^2 - \frac{9}{16} + 3} = \frac{1}{2}\int \frac{\mathrm{d}x}{\left(x - \frac{3}{4}\right)^2 + \frac{39}{16}} =$$

$$\frac{16}{2 \cdot 39} \frac{\sqrt{39}}{4} \int \frac{1}{\left(\frac{4}{\sqrt{39}}(x - \frac{3}{4})\right)^2 + 1} \frac{4}{\sqrt{39}} \mathrm{d}x =$$

$$= \frac{1}{2}\frac{4}{\sqrt{39}}\arctan \frac{4}{\sqrt{39}}\left(x - \frac{3}{4}\right) + c = \frac{2}{\sqrt{39}}\arctan \frac{4x - 3}{\sqrt{39}} + c.$$

If the numerator contains a linear function, it is necessary to express the fraction as the sum of two fractions where the numerator of one of them is equal to the derivative of the denominator (this leads to the logarithm) and the numerator of the other one is a constant (this leads to arcus tangent):

☛ ***Example 1.9.***

Find the integral $I = \int \frac{3x+5}{2x^2-3x+6}\,\mathrm{d}x\,.$

***Solution.***

$$I = \frac{3}{4}\int \frac{4x-3}{2x^2-3x+6}\,\mathrm{d}x + \int \frac{\frac{3}{4}\cdot 3+5}{2x^2-3x+6}\,\mathrm{d}x =$$

$$= \frac{3}{4}\ln\left|2x^2-3x+6\right| + \frac{33}{4}\cdot\frac{2}{\sqrt{39}}\arctan\frac{4x-3}{\sqrt{39}} + c.$$

To find the integral

$$K_n = \int \frac{\mathrm{d}x}{(x^2+a^2)^n}\,, \quad a \in \mathbb{R},\ a \neq 0,\ n \in \mathbb{N},\ x \in \mathbb{R},$$

we use the recursive formula

$$K_{n+1} = \frac{x}{2na^2(x^2+a^2)^n} + \frac{2n-1}{2na^2} K_n, \quad n = 1, 2, \ldots, x \in \mathbb{R}\,, \tag{1.1}$$

that is the result of the use of per partes:

$$K_n = \int 1 \cdot \frac{1}{(x^2+a^2)^n}\,\mathrm{d}x = \left| \begin{array}{ll} u' = 1, & u = x \\ v = \dfrac{1}{(x^2+a^2)^n}, & v' = -\dfrac{2nx}{(x^2+a^2)^{n+1}} \end{array} \right| =$$

$$= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{x^2}{(x^2+a^2)^{n+1}}\,\mathrm{d}x = \frac{x}{(x^2+a^2)^n} + 2n \int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}\,\mathrm{d}x =$$

$$= \frac{x}{(x^2+a^2)^n} + 2n \int \left[ \frac{1}{(x^2+a^2)^n} - a^2 \frac{1}{(x^2+a^2)^{n+1}} \right] \mathrm{d}x = \frac{x}{(x^2+a^2)^n} + 2nK_n$$

This follows (1.1).

# Substitutions leading to rational functions

Using the second theorem on the substitution, various integrals can be transformed to integrals of rational functions.

# The integrand contains goniometric functions

**In integrals of the type**

$$\int R(\sin^2 x, \cos x) \sin x \, \mathrm{d}x \,,$$

we can use the substitution $t = \cos x$, $\mathrm{d}t = -\sin x \, \mathrm{d}x$ that leads to a rational function of the variable $t$.

☛ ***Example***. Find the integral $\int \sin^3 x \, \mathrm{d}x$.

***Solution.***

$$\int \sin^3 x \, \mathrm{d}x = \int \sin^2 x \sin x \, \mathrm{d}x = \int (1 - \cos^2 x) \sin x \, \mathrm{d}x =$$

$$-\int (1 - t^2) \, \mathrm{d}t = t^3/3 - t + c = (\cos^3 x)/3 - \cos x + c.$$

**In integrals of the type**

$$\int R(\sin x, \cos^2 x) \cos x \, \mathrm{d}x \,,$$

we can use the substitution $t = \sin x,\ \mathrm{d}t = \cos x \, \mathrm{d}x$ that leads to a rational function of the variable $t$.

Similarly for integrals

$$\int \sin^m x \cos^n x \, \mathrm{d}x, \quad \text{where} \quad m, n \in \mathbb{N},$$

where at least one of the numbers $m, n$ is odd. If both numbers are even, we can use the relations

$$\sin^2 x = (1 - \cos 2x)/2; \quad \cos^2 x = (1 + \cos 2x)/2, \tag{1.2}$$

(eventually repeatedly).

☛ ***Example***

$$\int \sin^2 x \, \mathrm{d}x = \int (1 - \cos 2x)/2 \, \mathrm{d}x = x/2 - (\sin 2x)/4 + c.$$

☛ ***Example***

$$\int \cos^4 x \, \mathrm{d}x = \int (\cos^2 x)^2 \, \mathrm{d}x = (1/4) \int (1 + \cos 2x)^2 \, \mathrm{d}x =$$

$$= (1/4) \int (1 + 2\cos 2x + \cos^2 2x) \, \mathrm{d}x =$$

$$= x/4 + (\sin 2x)/4 + (1/4) \int (\cos^2 2x) \, \mathrm{d}x =$$

$$= x/4 + (\sin 2x)/4 + (1/8) \int (1 + \cos 4x) \, \mathrm{d}x =$$

$$= 3x/8 + (\sin 2x)/4 + (\sin 4x)/32 + c.$$

☛ ***Example***

$$\int \frac{\sin^3 x}{\cos^2 x + 1}\,\mathrm{d}x = \int \frac{\sin^2 x}{1+\cos^2 x}\sin x\,\mathrm{d}x =$$

$$-\int \frac{1-z^2}{1+z^2}\,\mathrm{d}z = z - 2\arctan z + c = \cos x - 2\arctan(\cos x) + c.$$

☛ ***Example***

$$\int \frac{\mathrm{d}x}{\sin x} = \int \frac{\mathrm{d}x}{\sin x} = \int \frac{\sin x}{\sin^2 x}\,\mathrm{d}x = \int \frac{\sin x}{1-\cos^2 x}\,\mathrm{d}x =$$

$$\cdots = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + c.$$

**In integrals of the type**

$$\int R(\sin^2 x, \cos^2 x, \sin x \cos x)\,\mathrm{d}x\,,$$

we can use the substitution $t = \tan x$ that leads to a rational function of the variable $t$.

Obviously,

$$\begin{aligned}
x &= \arctan t, \qquad \mathrm{d}x = \frac{1}{1+t^2}\,\mathrm{d}t \\
\cos^2 x &= \frac{\cos^2 x}{\cos^2 x + \sin^2 x} = \frac{1}{1+\tan^2 x} = \frac{1}{1+t^2} \\
\sin^2 x &= \frac{\sin^2 x}{\cos^2 x + \sin^2 x} = \frac{\tan^2 x}{1+\tan^2 x} = \frac{t^2}{1+t^2} \\
\sin x \cos x &= \frac{\sin x \cos x}{\cos^2 x + \sin^2 x} = \frac{\tan x}{1+\tan^2 x} = \frac{t}{1+t^2}.
\end{aligned} \tag{1.3}$$

☛ ***Example***

$$\int \tan^4 x \,\mathrm{d}x = \left| t = \tan x, \;\; \mathrm{d}x = \frac{1}{1+t^2}\,\mathrm{d}t \right| = \int \frac{t^4 - 1 + 1}{1+t^2}\,\mathrm{d}t =$$

$$= \int (t^2 - 1 + \frac{1}{1+t^2})\,\mathrm{d}t = \frac{1}{3}t^3 - t + \arctan t + c = \frac{1}{3}\tan^3 x - \tan x + x$$

☛ ***Example.*** $\displaystyle\int \frac{\mathrm{d}x}{\sin^2 x - 4\sin x \cos x + 5\cos^2 x}$.

***Solution.*** For $x \neq \pi/2 + k\pi$, $k$ integral,

$$\int \frac{\mathrm{d}x}{\sin^2 x - 4\sin x\cos x + 5\cos^2 x} = \int \frac{1}{\tan^2 x - 4\tan x + 5}\,\frac{1}{\cos^2 x}\,\mathrm{d}$$

$$= \int \frac{\mathrm{d}z}{z^2 - 4z + 5} = \int \frac{\mathrm{d}z}{(z-2)^2 + 1} = \arctan(z-2) + c =$$

$$= \arctan(\tan x - 2) + c.$$

## Universal substitution for $\int R(\sin x, \cos x)\,\mathrm{d}x$

The substitution $t = \tan\frac{x}{2}$ brings any integral of a rational function of $\sin x$ and $\cos x$ to the integral of a rational function of the variable $t$, namely on any interval that does not contain $(2k+1)\pi$, $k \in \mathbb{Z}$.

$$t = \tan\tfrac{x}{2} \quad\Longrightarrow\quad x = 2\arctan t, \qquad \mathrm{d}x = \frac{2}{1+t^2}$$

$$t^2 = \tan^2\frac{x}{2} = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1-\cos x}{1+\cos x}$$

$$t^2(1+\cos x) = 1-\cos x \quad\Longrightarrow\quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\sin^2 x = 1-\cos^2 x = 1-\frac{(1-t^2)^2}{(1+t^2)^2} = \frac{4t^2}{(1+t^2)^2} \quad\Longrightarrow\quad \sin x = \frac{2t}{1+t^2}$$

The functions obtained in this way are often complicated and if it is possible, it is more convenient to use some of the previous substitutions.

☛ ***Example***. Find the integral $\int \frac{1-\sin x}{1+\cos x}\,\mathrm{d}x,\ x \neq (2k+1)\pi$.

$$\int \frac{1-\sin x}{1+\cos x}\,\mathrm{d}x = \left| \begin{array}{ll} t = \tan\frac{x}{2}, & \sin x = \frac{2t}{1+t^2} \\ \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t, & \cos x = \frac{1-t^2}{1+t^2} \end{array} \right| =$$

$$= \int \frac{1-\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,\mathrm{d}t = \int \frac{t^2-2t+1}{1+t^2}\,\mathrm{d}t =$$

$$= \int \frac{1+t^2}{1+t^2}\,\mathrm{d}t - \int \frac{2t}{1+t^2}\,\mathrm{d}t = t - \ln(1+t^2) + c =$$

$$= \tan\frac{x}{2} - \ln\left(1+\tan^2\frac{x}{2}\right) + c, \quad x \neq (2k+1)\pi.$$

**Integrals of the type** $\int \sin ax\ \cos bx\,\mathrm{d}x$, $\int \sin ax\ \sin bx\,\mathrm{d}x$, $\int \cos ax\ \cos bx\,\mathrm{d}x$, $a \neq b$, can be found with the help of the relations

$$\begin{aligned} \sin\alpha\ \cos\beta &= (\sin(\alpha+\beta)+\sin(\alpha-\beta))/2, \\ \sin\alpha\ \sin\beta &= (\cos(\alpha-\beta)-\cos(\alpha+\beta))/2, \\ \cos\alpha\ \cos\beta &= (\cos(\alpha+\beta)+\cos(\alpha-\beta))/2. \end{aligned} \tag{1.4}$$

☛ ***Example***

$$\int \sin 5x \cos x\,\mathrm{d}x = (1/2)\int (\sin 6x + \sin 4x)\,\mathrm{d}x =$$

$$= -(1/12)\cos 6x - (1/8)\cos 4x + c.$$

**In integrals of the type** $\int R(e^{\alpha x})\,\mathrm{d}x$, $\alpha \in R$, $\alpha \neq 0$, we can use the substitution

$$e^{\alpha x} = t, \quad x = \frac{1}{\alpha}\ln t, \quad \mathrm{d}x = \frac{1}{\alpha t}\,\mathrm{d}t$$

leading to a rational function $\displaystyle \frac{1}{\alpha}\int R(t)\frac{\mathrm{d}t}{t}$.

☛ ***Example***

$$\int \frac{\mathrm{d}x}{(1+e^x)^2} = \int \frac{\mathrm{d}t}{t(1+t)^2} = \ln\left|\frac{t}{1+t}\right| + \frac{1}{1+t} + c =$$

$$= \ln\left|\frac{e^x}{1+e^x}\right| + \frac{1}{1+e^x} + c$$

☛ ***Example***

$$\int \frac{e^x\,\mathrm{d}x}{1+e^{2x}} = \int \frac{\mathrm{d}t}{1+t^2} = \arctan t + c = \arctan e^x + c$$

**In integrals of the type**

$$\int R(\ln x)\frac{\mathrm{d}x}{x} \tag{1.5}$$

we can use the substitution $\ln x = t,\ \mathrm{d}t = \dfrac{\mathrm{d}x}{x}$ that leads to the integral of a rational function $\int R(t)\,\mathrm{d}t$.

☛ ***Example***

$$\int \frac{\ln x}{1+\ln x}\frac{\mathrm{d}x}{x} = \left| \begin{array}{c} \ln x = t \\ \mathrm{d}t = \dfrac{\mathrm{d}x}{x} \end{array} \right| = \int \frac{t}{1+t}\,\mathrm{d}t =$$

$$= t - \ln|t+1| + c = \ln x - \ln|1+\ln x| + c.$$

## Integrand contains roots of a quotient of linear functions

In integrals of the type

$$\int R\left(x,\ \sqrt[n]{\frac{ax+b}{cx+e}}\right)\,\mathrm{d}x,\ n\in\mathbb{N},\ a,b,c,e\in\mathbb{R},\ ae-bc\neq 0,$$

we can use the substitution

$$t^n=\frac{ax+b}{cx+e},\quad x=\frac{et^n-b}{a-ct^n},\quad \mathrm{d}x=\frac{(ae-bc)nt^{n-1}}{(a-ct^n)^2}\,\mathrm{d}t.$$

If $n$ is even, it is necessary to suppose that $(ax+b)/(cx+e)\geq 0$, since $t^n\geq 0$. Then

$$\int R\left(x,\ \sqrt[n]{\frac{ax+b}{cx+e}}\right)\,\mathrm{d}x=\int R\left(\frac{et^n-b}{a-ct^n},t\right)\frac{(ae-bc)nt^{n-1}}{(a-ct^n)^2}\,\mathrm{d}t.$$

The integrand on the right side is a rational function of a variable

☛ ***Example***

$$I = \int \sqrt[3]{\frac{2x+1}{x+1}}\,\frac{\mathrm{d}x}{x^2} = \left| t^3 = \frac{2x+1}{x+1},\ x = \frac{t^3-1}{2-t^3},\ \mathrm{d}x = \frac{3t^2}{(2-t^3)^2}\,\mathrm{d}t \right| =$$

$$= \int t \left(\frac{2-t^3}{t^3-1}\right)^2 \frac{3t^2}{(2-t^3)^2}\,\mathrm{d}t = 3\int \frac{t^3}{(t^3-1)^2}\,\mathrm{d}t.$$

This integral can easily be calculated (try it!). The result is

$$I = \frac{t}{1-t^3} + \frac{1}{3}\ln|t-1| - \frac{1}{6}\ln(t^2+t+1) - \frac{1}{\sqrt{3}}\arctan\frac{2t+1}{\sqrt{3}} + c,$$

where $t$ must be replaced by $t = \sqrt[3]{\frac{2x+1}{x+1}}$.

### ☛ *Example*

$$I = \frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}} = \frac{\left[\sqrt{1-\sqrt{x}}\right]^2}{\sqrt{1+\sqrt{x}}\sqrt{1-\sqrt{x}}} = \frac{1-\sqrt{x}}{\sqrt{1-x}} =$$

$$= \frac{1}{\sqrt{1-x}} - \sqrt{\frac{x}{1-x}}$$

$$I_1 = \int \frac{1}{\sqrt{1-x}}\,\mathrm{d}x = \int (1-x)^{-1/2}\,\mathrm{d}x = -2(1-x)^{1/2}+c = -2\sqrt{1-x}+$$

$$I_2 = \int \sqrt{\frac{x}{1-x}}\,\mathrm{d}x = \left| \begin{array}{l} \sqrt{\frac{x}{1-x}} = t,\ x = \frac{t^2}{1+t^2} \\ \mathrm{d}x = \frac{2t}{(1+t^2)^2}\,\mathrm{d}t \end{array} \right| = \int \frac{2t^2}{(1+t^2)^2}\,\mathrm{d}t.$$

We obtain $I_2 = \arctan t - t/(1+t^2) + c$, where $t = \sqrt{x/(1-x)}$, thus

$$I = I_1 - I_2 = -2\sqrt{1-x} - \arctan\sqrt{\frac{x}{1-x}} - \sqrt{x(1-x)} + c.$$

☛ ***Example***

$$\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\,\mathrm{d}x =$$

$$= \int \frac{(\sqrt{x+1}-\sqrt{x-1})^2}{2}\,\mathrm{d}x = \frac{1}{2}\int (2x-2\sqrt{(x+1)(x-1)})\,\mathrm{d}x =$$

$$= \int (x-\sqrt{(x+1)(x-1)})\,\mathrm{d}x = \frac{x^2}{2} - \int \sqrt{\frac{x+1}{x-1}}(x-1)\,\mathrm{d}x =$$

$$= \left| t^2 = \frac{x+1}{x-1},\ x = \frac{t^2+1}{t^2-1},\ x-1 = \frac{2}{t^2-1},\ \mathrm{d}x = \frac{-4t}{(t^2-1)^2}\,\mathrm{d}t \right| =$$

$$= \frac{x^2}{2} + 4\int t\frac{2}{t^2-1}\,\frac{t}{(t^2-1)^2}\,\mathrm{d}t = \frac{x^2}{2} + 8\int \frac{t^2}{(t^2-1)^3}\,\mathrm{d}t,$$

which is an integral of a rational function.