LECTURE 3

DEFINITE INTEGRAL

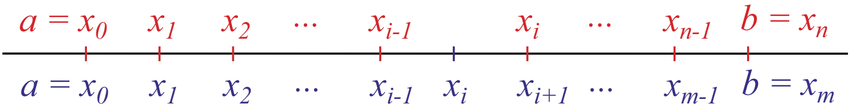
Riemann Integral

Definition 1. Let $I = [a, b]$ be a bounded interval in $\mathbb{R}$.
A partition D of $[a, b]$ is any finite sequence of points

$$x_0 = x \leq x_1 \leq x_2 \leq \cdots < x_{n-1} \leq x_n = b\,.$$

Definition 2. A partition D' of an interval $[a, b]$ is **a refinement of the partition D** if $D \subset D'$.

Partition D:



Partition D' - a refinement of the partition D

Definition 3. Let f be a function that is bounded on an interval $[a, b]$. Denote

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \qquad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x).$$

For any partition D of $[a, b]$, the numbers

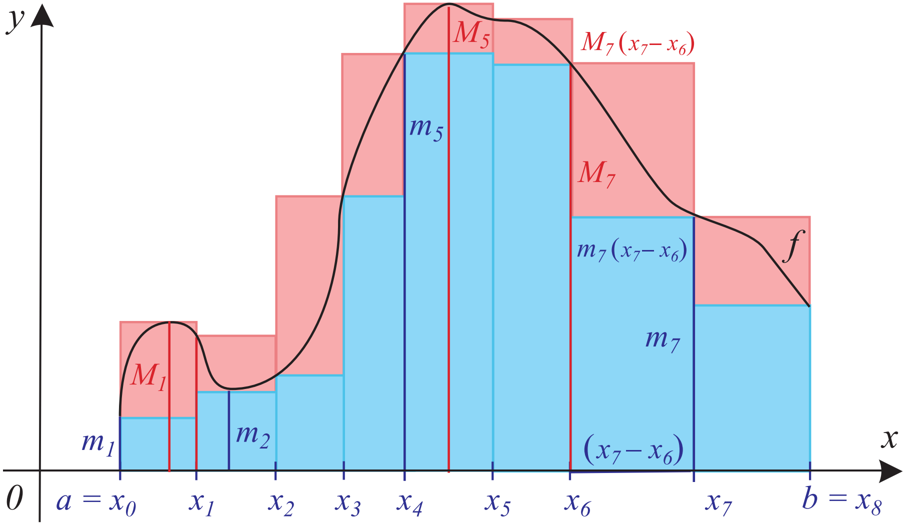
$$s_D = \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1}), \qquad S_D = \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1}).$$

are called the **lower Riemann sum** and the S_D **upper Riemann sum of $f(x)$ corresponding to the partition D**.

Remark. Since $m_i \leq M_i$ for any i, it is

$$s_D \leq S_D.$$

Geometric interpretation



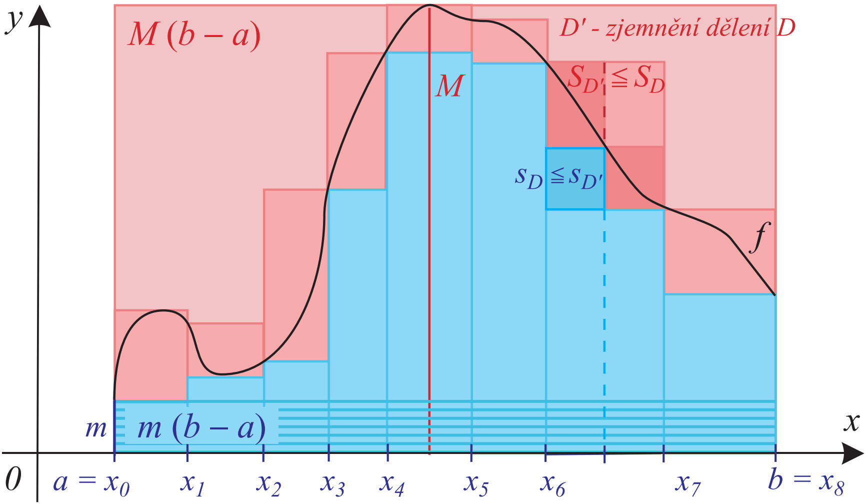
The lower sum represents the area of the inscribed region composed by rectangles that are all bellow the graph, but touch it. For any partition, this area is lower or equal to the area of the region circumscribed by the axis x and the graph of the function on the given interval. Similarly, the upper sum gives the area of the circumscribed region and is always greater or equal to the area of the region between the axis x and the graph of the function.

Denote by $\mathcal{D}$ the set of all partitions of the interval $[a, b]$,

$$M = \sup_{x \in [a,b]} f(x) , \qquad m = \inf_{x \in [a,b]} f(x) .$$

Theorem. *If D' is a refinement of a partition D, the following inequalities are satisfied:*

$$m(b-a) \leq s_D \leq s_{D'} \leq S_{D'} \leq S_D \leq M(b-a)\,.$$



Theorem. *For any two partitions D_1 and D_2 of the interval $[a, b]$, the following inequality is satisfied: $s_{D_1} \leq S_{D_2}$.*

The set $\{s_D; D \in \mathcal{D}\}$ is **bounded from above**, e.g., by the number $M(b-a)$, and **bounded from below**, e.g., by $m(b-a)$. Thus there exist

$$s = \sup_{D \in \mathcal{D}} s_D \in \mathbb{R} \qquad \text{and} \qquad S = \inf_{D \in \mathcal{D}} S_D \in \mathbb{R}.$$

Definition 4. The number $s = \sup_{D \in \mathcal{D}} s_D$ is called **the lower Riemann integral of the function f on the interval $[a, b]$**, the number $S = \inf_{D \in \mathcal{D}} S_D$ is called **the upper Riemann integral of the function f on the interval $[a, b]$**.

Definition 5. Let $I = [a, b]$ be a bounded interval in $\mathbb{R}$, let $f(x)$ be a function that is bounded on I. If $s = S$, where s and S are the lower and upper Riemann integrals of $f(x)$ on I, this common value is called **the Riemann integral of the function $f(x)$ on the interval $I = [a, b]$** and it is denoted by the symbol $\int_a^b f(x)\,\mathrm{d}x$. In this case we say that the function f is **integrable** on I.

The Riemann integral is also called **the definite integral of f on** $[a, b]$.

Remark. Notice that the Riemann integral is defined only for bounded functions on bounded intervals.

Theorem. *If a function f is monotonous on an interval $[a,b]$, then it is integrable on $[a,b]$.*

Theorem. *If a function f is continuous on an interval $[a,b]$, then it is integrable on $[a,b]$.*

Theorem. *A function f is integrable on an interval $[a,b]$ if and only if it is integrable on the intervals $[a,c]$ and $[c,b]$ for any $c \in (a,b)$. Moreover, it is*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x\,.$$

Remark. The Riemann integral $\int_a^b f(x)\,\mathrm{d}x$ was defined for $b \geq a$. The previous theorem allows to extend the definition to cover the case $b \leq a$, too. We can simply put

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x\,.$$

Theorem. *Let the functions* f_1 , f_2 *be integrable on an interval* $[a, b]$, *let* c_1, c_2 *be real constants. Then*

$$\int_a^b \big(c_1 f_1(x) + c_2 f_2(x)\big)\,\mathrm{d}x = c_1 \int_a^b f_1(x)\,\mathrm{d}x + c_2 \int_a^b f_2(x)\,\mathrm{d}x\,.$$

Theorem. *Let the function* f *be integrable on an interval* $[a, b]$, *let* $k \le f(x) \le K$ *for all* $x \in [a, b]$. *Then*

$$k(b-a) \le \int_a^b f(x)\,\mathrm{d}x \le K(b-a).$$

Theorem. *If* $f(x) \ge 0$ *is integrable on* $[a, b]$, *then* $\displaystyle\int_a^b f(x)\,\mathrm{d}x \ge 0$.

Theorem. *If* f, g *are integrable on* $[a, b]$ *and* $f(x) \le g(x)$ *for all* $x \in [a, b]$, *then* $\displaystyle\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x\,.$

Theorem. *If* f *is integrable on* $[a, b]$, *then*

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b \big|f(x)\big|\,\mathrm{d}x.$$

Integral as a function of the upper limit

Let f be a function that is integrable on an interval $[a, b]$. Then for any $x \in [a, b]$, there exists the integral $\int_a^x f(t)\,\mathrm{d}t$. Since the value of this integral is uniquely determined, we can define a function $F\colon [a, b] \to \mathbb{R}$ to be

$$F(x) = \int_a^x f(t)\,\mathrm{d}t\,, \; x \in [a, b]\,.$$

The integral in this equality is therefore **the function of the upper limit**. Analogously we can consider the integral that is **the function of the lower limit,** i.e.,

$$G(x) = \int_x^b f(t)\,\mathrm{d}t\,, \; x \in [a, b]\,.$$

For $x,\; x + h \in [a, b]$ it is

$$F(x+h) = \int_a^{x+h} f(t)\,\mathrm{d}t = \int_a^x f(t)\,\mathrm{d}t + \int_x^{x+h} f(t)\,\mathrm{d}t = F(x) + \int_x^{x+h} f(t)\,\mathrm{d}t.$$

Theorem. *Let f be a function that is integrable on $[a, b]$. Then the function $F(x) = \int\limits_a^x f(t)\,\mathrm{d}t$, $x \in [a, b]$ has the following properties:*

(i) *it is continuous on $[a, b]$;*

(ii) *if f is continuous at $x_0 \in (a, b)$, then $F'(x_0) = f(x_0)$. If $x_0 = a$ or $x_0 = b$, respectively, then $F'(a+) = f(a+)$ or $F'(b-) = f(b-)$, respectively;*

Newton–Leibniz formula

Theorem. *Let the function f be continuous on an interval $[a, b]$, let F be its primitive function. Then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Remark. Newton–Leibniz formula is usually denoted in the form

$$\int\limits_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = [F(x)]_a^b.$$

Obviously:

$$[F(x) \pm G(x)]_a^b = [F(x)]_a^b \pm [G(x)]_a^b, \qquad [cF(x)]_a^b = c[F(x)]_a^b$$

for any two functions F, G and any real number c.

Newton–Leibniz formula is very useful since it finaly provides the way how to find the Riemann integral. Moreover, it allows to use all integration methods that we have learned for indefinite integrals. As soon as we find a primitive function F for a given function f, it is sufficient to use the Newton-Leibniz formula.

Remark. Notice that the result is the same for all primitive function. If F, G are primitive functions to f on $[a, b]$, then there exists a real constant C such that $G(x) = F(x) + C$ for all $x \in [a, b]$, thus

$$G(b) - G(a) = (F(b) + C) - (F(a) + C) = F(b) - F(a)\,.$$

Definition 6. Let the function $f(x)$ be defined on an interval $[a, b]$. If $F(x)$ is a primitive function to $f(x)$ on $[a, b]$, the number $F(b) - F(a)$ is called **the Newton definite integral** of $f(x)$ on $[a, b]$.

☛ ***Example 3.1.***

Consider a function

$$f(x)=\begin{cases}\dfrac{1}{n} & \text{for } x=\dfrac{m}{n}\in\mathbb{Q},\ n\in\mathbb{N},\ m\in\mathbb{Z} \text{ have no common divisor}\\[2ex] 0 \text{ for } x\in\mathbb{R}\setminus\mathbb{Q}.\end{cases}$$

It can be proved that there exists the Riemann integral

$$\int_0^1 f(x)\,\mathrm{d}x=0.$$

But there does not exist a primitive function on $[0,1]$. Thus there does not exist the Newton integral.

3.1 Integration by parts (per partes)

Theorem. *Let the functions f, g have continuous derivatives on an interval $[a, b]$. Then:*

$$\int_a^b f'(x)g(x)\,\mathrm{d}x = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,\mathrm{d}x\,.$$

Proof. The functions

$$F(x) = \int_a^x f'(t)g(t)\ \mathrm{d}t\,, \quad \tilde{F}(x) = f(x)g(x) - f(a)g(a) - \int_a^x f(t)g'(t)$$

are primitive to $f'(x)g(x)$ on $[a, b]$. They can therefore differ only by an additive constant. But for $x = a$ it is $F(a) = \tilde{F}(a) = 0$. Thus for any $x \in [a, b]$, the following equality is satisfied:

$$\int_a^x f'(t)g(t)\ \mathrm{d}t = f(x)g(x) - f(a)g(a) - \int_a^x f(t)g'(t)\ \mathrm{d}t\,.$$

For $x = b$ we obtain the proposition. $\square$

3.2 Substitution in the Riemann integral

Theorem. *Let the function $f(x)$ be continuous on an interval $[a, b]$, let $\varphi : [\alpha, \beta] \to [a, b]$ be a continuous differentiable function, let $\varphi(\alpha) = a$, $\varphi(\beta) = b$. Then*

$$\int_{\alpha}^{\beta} f\big(\varphi(t)\big) \cdot \varphi'(t)\,\mathrm{d}t = \int_a^b f(x)\,\mathrm{d}x\,.$$

Proof. If $F(x)$ is a primitive function to $f(x)$ on $[a, b]$, then according to the theorem about the substitution in an indefinite integral, $\Psi(t) = F\big(\varphi(t)\big)$ is a primitive function to $f(\big(\varphi(t)\big) \cdot \varphi'(t)$ on $[\alpha, \beta]$. Thus

$$\int_{\alpha}^{\beta} f\big(\varphi(t)\big) \cdot \varphi'(t)\,\mathrm{d}t = \Psi(\beta) - \Psi(\alpha) = F\big(\varphi(\beta)\big) - F\big(\varphi(\alpha)\big) =$$

$$= F(b) - F(a) = \int_a^b f(x)\,\mathrm{d}x\,.$$

Theorem. *Let the function* $f(x)$ *be continuous on an interval* $[a,b]$, *let* $\varphi : [\alpha,\beta] \to [a,b]$ *be a continuous differentiable function that maps the interval* $[\alpha,\beta]$ *on* $[a,b]$, *let* $\varphi'(t) \neq 0$. *Then*

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f\big(\varphi(t)\big)\cdot\varphi'(t)\,\mathrm{d}t\,.$$

Proof. Let $\Psi(t)$ be a primitive function to $f\big(\varphi(t)\big)\cdot\varphi'(t)$ on $[\alpha,\beta]$. Then the integral on the right side is equal to $\Psi(\beta) - \Psi(\alpha)$. The assumptions of the theorem guarantee that there exists an inverse function $t = \varphi^{-1}(x)$ to the function $x = \varphi(t)$ and $F(x) = \Psi\big(\varphi^{-1}(x)\big)$ is a primitive function to $f(x)$. Thus

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = \Psi\big(\varphi^{-1}(b)\big) - \Psi\big(\varphi^{-1}(a)\big) =$$

$$= \Psi(\beta) - \Psi(\alpha) = \int_\alpha^\beta f\big(\varphi(t)\big)\cdot\varphi'(t)\,\mathrm{d}t\,.$$

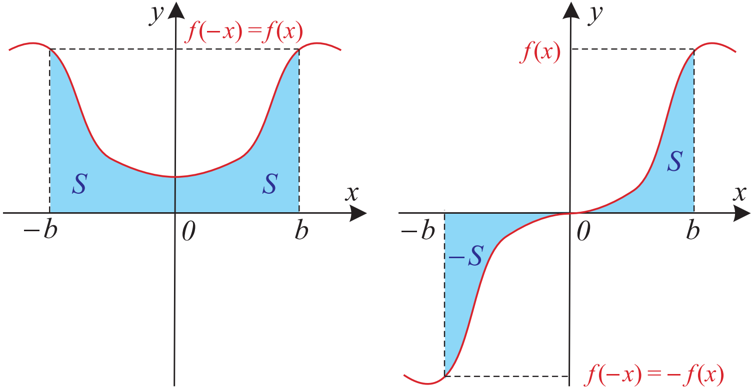
Integral of an even, odd or periodic function

Theorem. *If f is an even integrable function on $[-b, b]$, then*

$$\int_{-b}^{b} f(x)\,\mathrm{d}x = 2\int_{0}^{b} f(x)\,\mathrm{d}x.$$

Theorem. *If f is an odd integrable function on $[-b, b]$, then:*

$$\int_{-b}^{b} f(x)\,\mathrm{d}x = 0.$$



Theorem. *Let f be a periodic function with the period T, let a, a' be real numbers. If one of the following integrals exists, then the other one exists, too, and it is*

$$\int_{a}^{a+T} f(x)\,\mathrm{d}x = \int_{a'}^{a'+T} f(x)\,\mathrm{d}x.$$