LECTURE 9

LINE INTEGRALS OF THE FIRST KIND

Line integral of the first kind

Definition 1. The set $\mathcal{C} \subset \mathbb{R}^n$ is called **a simple regular curve** in $\mathbb{R}^n$ if there exists a 1-1 mapping

$$\boldsymbol{g}\colon (a,b) \to \mathcal{C}\,; \quad \boldsymbol{g}(t) = (g_1(t), \ldots, g_n(t))\,, \tag{9.1}$$

that has a continuous derivative

$$\dot{\boldsymbol{g}}\,(t) = (g_1'(t), \ldots, g_n'(t)) \neq (0, \ldots, 0)$$

on the interval (a,b).

It is therefore the set of points $\boldsymbol{x} = \boldsymbol{g}(t)$, i.e.,

$$(x_1, \ldots, x_n) = (g_1(t), \ldots, g_n(t))\,,$$

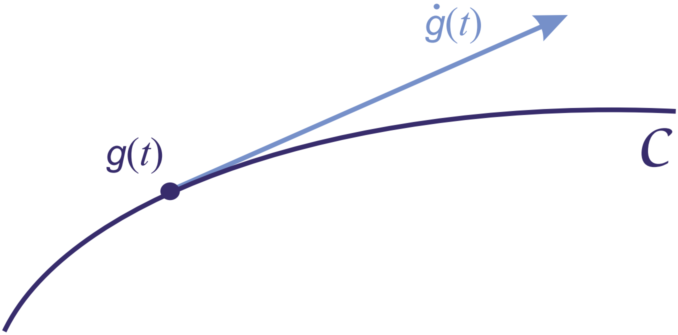
where the mapping $\boldsymbol{g}$ satisfies the contitions stated above. This mapping is called **a parametrization of the curve $\mathcal{C}$**, the equation $\boldsymbol{x} = \boldsymbol{g}(t)$ is called **a parametric equation of the curve $\mathcal{C}$**.

The vector

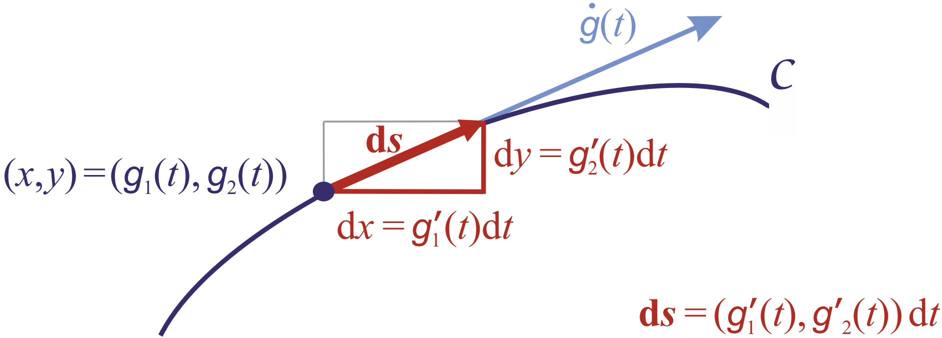
$$\dot{\boldsymbol{g}}(t) = (g_1'(t), \ldots, g_n'(t))$$

is a **tangent vectorem to the curve $\mathcal{C}$.**

We can imagine a curve as a trajectory of a point that is moving in time. Then $\boldsymbol{g}(t)$ gives the coordinates of this point in time t, the vector $\dot{\boldsymbol{g}}(t)$ gives its instantaneous velocity.



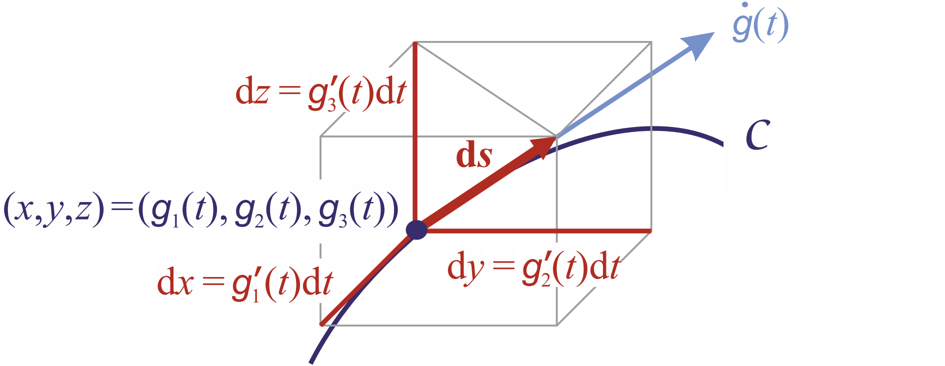
Special case: curves in $\mathbb{R}^2$



Length element $\mathrm{d}s$ of a simple regular curve $\mathcal{C}$ with the parametrization $\boldsymbol{g}(t)$ is equal to

$$\mathrm{d}s = \|\dot{\boldsymbol{g}}(t)\|\,\mathrm{d}t = \sqrt{(g_1'(t))^2 + (g_2'(t))^2}\,\mathrm{d}t \qquad (9.2)$$

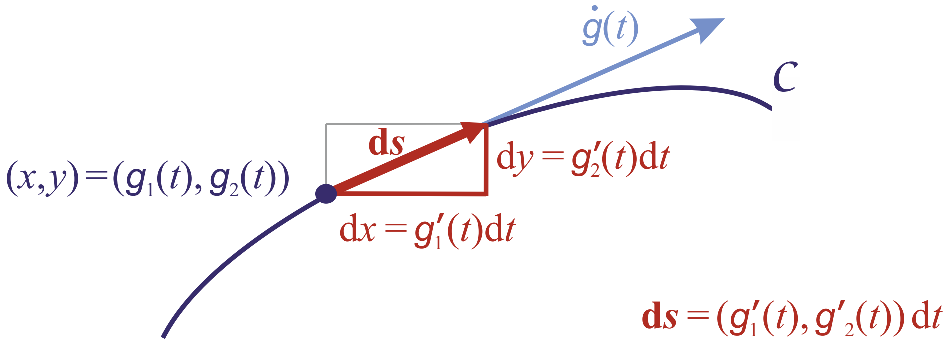
Special case: curves in $\mathbb{R}^3$



Length element $\mathrm{d}s$ of a simple regular curve $\mathcal{C}$ with the parametrization $\boldsymbol{g}(t)$ is equal to

$$\mathrm{d}s = \|\dot{\boldsymbol{g}}(t)\|\,\mathrm{d}t = \sqrt{(g_1'(t))^2 + (g_2'(t))^2 + (g_3'(t))^2}\,\mathrm{d}t \qquad (9.3)$$

Curves in $\mathbb{R}^n$



Length element $\mathrm{d}s$ of a simple regular curve $\mathcal{C}$ with the parametrization $\boldsymbol{g}(t)$ is equal to

$$\mathrm{d}s = \|\dot{\boldsymbol{g}}(t)\|\,\mathrm{d}t = \sqrt{(g_1'(t))^2 + \cdots + (g_n'(t))^2}\,\mathrm{d}t \qquad (9.4)$$

Line integral of the first kind

Definition 2. Let $\mathcal{C}$ be a simple regular curve in $\mathbb{R}^n$, let $\boldsymbol{g}\colon (a,b) \to \mathbb{R}$ be its parametrization. Let $f\colon \mathcal{C} \to \mathbb{R}$ be a function. If the Riemann integral

$$\int_a^b f(\boldsymbol{g}(t))\,\|\dot{\boldsymbol{g}}(t)\|\,\mathrm{d}t \tag{9.5}$$

exists, then it is denoted by

$$\int_{\mathcal{C}} f\,\mathrm{d}s \equiv \int_a^b f(\boldsymbol{g}(t))\,\|\dot{\boldsymbol{g}}(t)\|\,\mathrm{d}t \tag{9.6}$$

and called **the line integral of the first kind of the function f over the curve $\mathcal{C}$.**.

Properties

Linearity

If α, β are real numbers and f, g functions, then the equality

$$\int_{\mathcal{C}} (\alpha f + \beta h)\,\mathrm{d}s = \alpha \int_{\mathcal{C}} f\,\mathrm{d}s + \beta \int_{\mathcal{C}} h\,\mathrm{d}s \tag{9.7}$$

holds if the right side has a sense.

Additivity with respect to the curve

If $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2$ contains at most the boundary points, then the equality

$$\int_{\mathcal{C}} f\,\mathrm{d}s = \int_{\mathcal{C}_1 \cup \mathcal{C}_2} f\,\mathrm{d}s = \int_{\mathcal{C}_1} f\,\mathrm{d}s + \int_{\mathcal{C}_2} f\,\mathrm{d}s \tag{9.8}$$

holds, if the right side has a sense.

☛ ***Example***. Find the integral

$$\int_{\mathcal{C}} x^2 ds, \quad \text{kde} \quad \mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid y = \ln x, x \in \langle 1, 2 \rangle\}.$$

Solution.

Consider the parametrization $x = g_1(t) = t, y = g_2(t) = \ln t$.
Then $\dot{\boldsymbol{g}}(t) = (1, \frac{1}{t}) \neq \mathrm{o}$.

$$\int_{\mathcal{C}} x^2 \,\mathrm{d}s = \int_1^2 t^2 \left\| \left(1, \frac{1}{t}\right) \right\| \mathrm{d}t = \int_1^2 t^2 \sqrt{1 + \frac{1}{t^2}} \,\mathrm{d}t = \int_1^2 t\sqrt{t^2+1} \,\mathrm{d}t =$$

$$= \left| \begin{array}{lllllll} t^2+1 & = & u, & & 2t\,\mathrm{d}t & = & \mathrm{d}u \\ t & = & 1 & \Rightarrow & u & = & 2 \\ t & = & 2 & \Rightarrow & u & = & 5 \end{array} \right| =$$

$$= \frac{1}{2} \int_2^5 u^{1/2} \,\mathrm{d}u = \frac{1}{2} \cdot \frac{2}{3} [u^{3/2}]_2^5 = \frac{1}{3}(5^{3/2} - 2^{3/2}).$$

☛ ***Example***. Find the integral

$$\int_{\mathcal{C}} (x+y)ds,$$

where $\mathcal{C}$ is a line segment with the boundary points $A = (0,0)$, $B = (1,2)$.

Solution.

Parametrization: $x = t, y = 2t, \; t \in \langle 0,1 \rangle$. For this parametrization we obtain tangent vector $\dot{\boldsymbol{g}}(t) = (1,2)$ with $\|\dot{\boldsymbol{g}}(t)\| = \sqrt{5}$. Thus

$$\int_{\mathcal{C}} (x+y)\,\mathrm{d}s = \int_0^1 (t+2t)\sqrt{5}\,\mathrm{d}t = \frac{3}{2}\sqrt{5}.$$

☛ ***Example***. Find the integral

$$\int\limits_{\mathcal{C}} \frac{z^2}{x^2+y^2}\,\mathrm{d}s,$$

where $\mathcal{C}$ is one thread o the screw $x = r\cos t, y = r\sin t, z = rt,\ t \in \langle 0, 2\pi\rangle$.

Solution. Tangent vector:

$$\dot{\boldsymbol{g}}\,(t) = (-r\sin t, r\cos t, r)$$

is size: $||\dot{\boldsymbol{g}}\,(t)|| = r\sqrt{2}$. Then

$$\int\limits_{\mathcal{C}} \frac{z^2}{x^2+y^2}\,\mathrm{d}s = \int\limits_0^{2\pi} \frac{r^2t^2}{r^2} r\sqrt{2}\,\mathrm{d}t = \frac{8r\pi^3\sqrt{2}}{3}.$$

Some applications

Length $s(\mathcal{C})$ of a curve $\mathcal{C}$:

$$s(\mathcal{C}) = \int_{\mathcal{C}} \mathrm{d}s = \int_{t_0}^{t_1} \|\dot{\boldsymbol{g}}(t)\| \, \mathrm{d}t$$

Weight $m(\mathcal{C})$ of a curve $\mathcal{C}$ with the length density σ :

$$m(\mathcal{C}) = \int_{\mathcal{C}} \sigma \, \mathrm{d}s = \int_{t_0}^{t_1} \sigma(\boldsymbol{g}(t)) \|\dot{\boldsymbol{g}}(t)\| \, \mathrm{d}t$$

Static moments in $\mathbb{R}^2$:

$$S_y(\mathcal{C}) = \int\limits_{\mathcal{C}} x\sigma \,\mathrm{d}s = \int\limits_{t_0}^{t_1} g_2(t)\sigma(\boldsymbol{g}(t))\|\dot{\boldsymbol{g}}(t)\|\,\mathrm{d}t$$

$$S_x(\mathcal{C}) = \int\limits_{\mathcal{C}} y\sigma \,\mathrm{d}s = \int\limits_{t_0}^{t_1} g_1(t)\sigma(\boldsymbol{g}(t))\|\dot{\boldsymbol{g}}(t)\|\,\mathrm{d}t$$

Coordinates $x_{cg}(\mathcal{C})$, $y_{cg}(\mathcal{C})$ of a centre of gravity of a curve $\mathcal{C}$:

$$x_{cg}(\mathcal{C}) = \frac{S_y(\mathcal{C})}{m(\mathcal{C})} = \frac{\int\limits_{\mathcal{C}} x\sigma \,\mathrm{d}s}{m(\mathcal{C})},$$

$$y_{cg}(\mathcal{C}) = \frac{S_x(\mathcal{C})}{m(\mathcal{C})} = \frac{\int\limits_{\mathcal{C}} y\sigma \,\mathrm{d}s}{m(\mathcal{C})}$$

Static moments in $\mathbb{R}^3$:

$$S_{yz}(\mathcal{C}) = \int\limits_{\mathcal{C}} x\sigma \,\mathrm{d}s = \int\limits_{t_0}^{t_1} g_1(t)\sigma(\boldsymbol{g}(t))\|\dot{\boldsymbol{g}}(t)\| \,\mathrm{d}t$$

$$S_{xz}(\mathcal{C}) = \int\limits_{\mathcal{C}} y\sigma \,\mathrm{d}s = \int\limits_{t_0}^{t_1} g_2(t)\sigma(\boldsymbol{g}(t))\|\dot{\boldsymbol{g}}(t)\| \,\mathrm{d}t$$

$$S_{xy}(\mathcal{C}) = \int\limits_{\mathcal{C}} z\sigma \,\mathrm{d}s = \int\limits_{t_0}^{t_1} g_3(t)\sigma(\boldsymbol{g}(t))\|\dot{\boldsymbol{g}}(t)\| \,\mathrm{d}t$$

Coordinates $x_{cg}(\mathcal{C})$, $y_{cg}(\mathcal{C})$, $y_{cg}(\mathcal{C})$ of the centre of gravity of $\mathcal{C}$:

$$x_{cg}(\mathcal{C}) = \frac{\int\limits_{\mathcal{C}} x\sigma \,\mathrm{d}s}{m(\mathcal{C})}, \quad y_{cg}(\mathcal{C}) = \frac{\int\limits_{\mathcal{C}} y\sigma \,\mathrm{d}s}{m(\mathcal{C})}, \quad z_{cg}(\mathcal{C}) = \frac{\int\limits_{\mathcal{C}} z\sigma \,\mathrm{d}s}{m(\mathcal{C})}.$$

Moments of inertia in $\mathbb{R}^2$:

$$I_x(\mathcal{C}) = \int_{\mathcal{C}} y^2 \sigma \, \mathrm{d}s = \int_{t_0}^{t_1} g_2^2(t) \sigma(\boldsymbol{g}(t)) \|\dot{\boldsymbol{g}}(t)\| \, \mathrm{d}t$$

$$I_y(\mathcal{C}) = \int_{\mathcal{C}} x^2 \sigma \, \mathrm{d}s = \int_{t_0}^{t_1} g_1^2(t) \sigma(\boldsymbol{g}(t)) \|\dot{\boldsymbol{g}}(t)\| \, \mathrm{d}t$$

Moments of inertia in $\mathbb{R}^3$:

$$I_x(\mathcal{C}) = \int_{\mathcal{C}} (y^2 + z^2)\sigma \, \mathrm{d}s = \int_{t_0}^{t_1} (g_2^2(t) + g_3^2(t))\sigma(\boldsymbol{g}(t)) \|\dot{\boldsymbol{g}}(t)\| \, \mathrm{d}t$$

$$I_y(\mathcal{C}) = \int_{\mathcal{C}} (x^2 + z^2)\sigma \, \mathrm{d}s = \int_{t_0}^{t_1} (g_2^2(t) + g_3^2(t))\sigma(\boldsymbol{g}(t)) \|\dot{\boldsymbol{g}}(t)\| \, \mathrm{d}t$$

$$I_z(\mathcal{C}) = \int_{\mathcal{C}} (x^2 + y^2)\sigma \, \mathrm{d}s = \int_{t_0}^{t_1} (g_1^2(t) + g_2^2(t))\sigma(\boldsymbol{g}(t)) \|\dot{\boldsymbol{g}}(t)\| \, \mathrm{d}t$$