CHAPTER 1

NUMBER SETS

Sets – basic knowledge

Set is any collection of objects. It is determined, if it is possible to decide unambiguously on each object whether it belongs to it or not. Each element belonging to a set is named its **element**.

Notation:

Set . . . **A**, **B**, **M**etc. (capital letters)

Their elements . . . a, b, x etc. (small letters)

a **is an element of a set A** . . . $a \in$ **A**

b **is not an element of a set A** . . . $b \notin$ **A**.

Empty set: $A = \emptyset$ . . . no element

Non-empty set: $A \neq \emptyset$ . . . at least one element

Finite set . . . number of elements given by a natural number

Infinite set . . . otherwise

Determining sets

List of elements

e.g., **A** $= \{1, 3, 5, 7\}$

Characteristic property

e.g., **A** $= \{x \in \mathbb{R};\ V(x)\}$

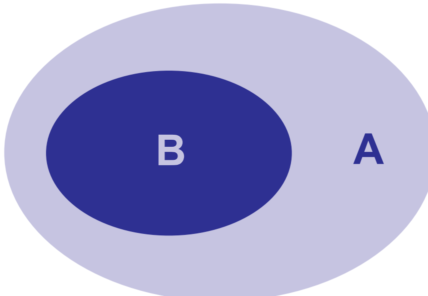
A is the set of all x from the set $\mathbb{R}$ for which $V(x)$ holds

(or: that have a property $V(x)$)

Set relations and operations

A, **B** . . . sets

B $\subset$ **A . . . B is a subset of A,** if each element of **B** belongs to **A**



☛ ***Example***

A $= \{1, 3, 5, 7, 9, 11, 13\}$, **B** $= \{3, 5, 11, 13\}$ . . . **B** $\subset$ **A**

A $=$ **B . . . sets A, B are called equal**, if **A** $\subset$ **B** and **B** $\subset$ **A**, i.e., they contain the same elements.

Union A $\cup$ B . . . the set of all elements that belong to A or B

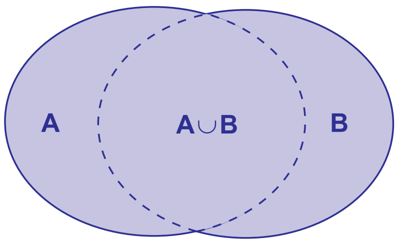
Obviously: $\mathbf{A} \cup \emptyset = \mathbf{A}$.

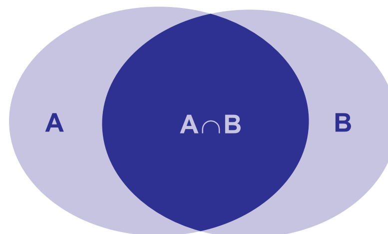
Intersection A $\cap$ B

. . . the set of all elements that belong to A and B

Obviously: $\mathbf{A} \cap \emptyset = \emptyset$

Disjoint sets A, B . . . A $\cap$ B $= \emptyset$

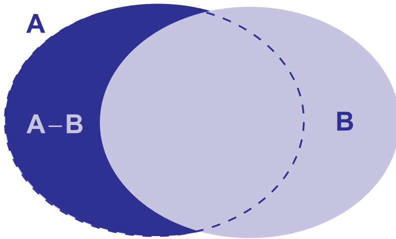


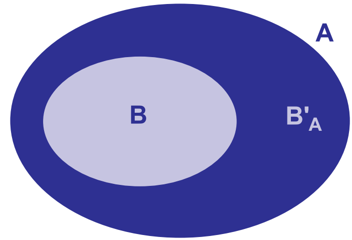


☛ ***Example***

- $\{2, 3, 4, 5, 6, 7, 8\} \cup \{1, 3, 5, 7, 9, 11, 13\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13\}$
- $\{2, \mathbf{3}, 4, \mathbf{5}, 6, \mathbf{7}, 8\} \cap \{1, \mathbf{3}, \mathbf{5}, \mathbf{7}, 9, 11, 13\} = \{\mathbf{3}, \mathbf{5}, \mathbf{7}\}$

Set difference $\mathbf{A} \setminus \mathbf{B}$
. . . the set of all elements of A that do not belong to B.
Complement of B in A . . . $\mathbf{B}^c = \mathbf{A} \setminus \mathbf{B}$, where $\mathbf{B} \subset \mathbf{A}$





☛ ***Example***

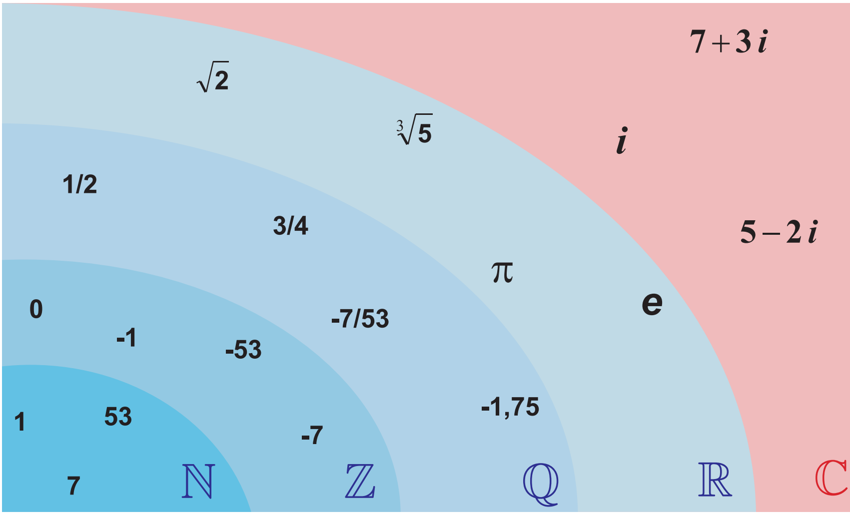
- *Difference of sets*:

$$\{2, 3, 4, 5, 6, 7, 8\} \setminus \{1, 3, 5, 7, 9, 11, 13\} = \{2, 4, 6, 8\}$$

- *Complement of a set*
 $\boldsymbol{B} = \{3, 5, 7, 9\}$ *in a set* $\boldsymbol{A} = \{1, 3, 5, 7, 9, 11, 13\}$:

$$\mathbf{B}'_{\mathbf{A}} = \{1, 11, 13\}.$$

Number sets

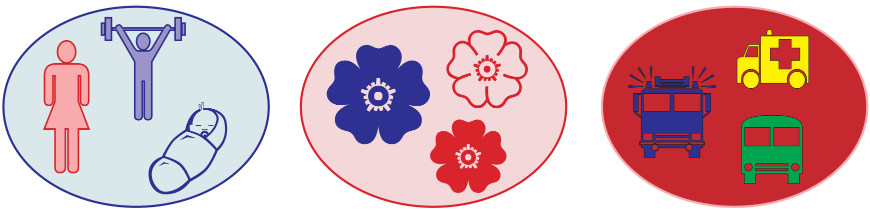


$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Natural numbers

$$\mathbb{N} = \{1, 2, 3, \dots\}, \qquad \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$$

Natural numbers allow to express the number of elements of finite non-empty sets. A natural number (consider e.g. 3) can be understood as a common property of the following sets:



The set of natural numbers is **closed with respect to addition and multiplication,** not with respect to subtraction (e.g., $2 - 5 = -3$) and division (e.g., $2 : 5 = 0.40$). In the domain of natural numbers, there exists neither **opposite** or **inverse** number to any natural number.

Positional decimal system

For example: $87\,956 =$

$$
\begin{aligned}
& 8 \cdot 10\,000 + 7 \cdot 1\,000 + 9 \cdot 100 + 5 \cdot 10 + 6 \cdot 1 \\
= \; & 8 \cdot 10^4 + 7 \cdot 10^3 + 9 \cdot 10^2 + 5 \cdot 10^1 + 6 \cdot 10^0 \,.
\end{aligned}
$$

In general: $a = a_n a_{n-1} \dots a_1 a_0$ means

$$a = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots + a_1 \cdot 10^1 + a_0 \cdot 10^0 \,,$$

where $n \in \mathbb{N}_0$, $a_n, a_{n-1}, \dots, a_1, a_0 \in \{0, 1, \dots, 9\}$, $a_n \neq 0$

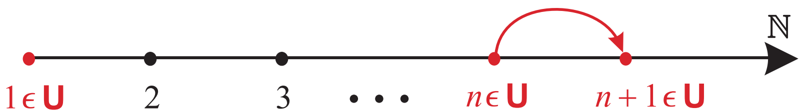
Axiom of mathematical induction.

Let $U \subset \mathbb{N}$ be a set such that

1) $1 \in U$,

2) if $n \in U$, then $(n+1) \in U$.

Then $U = \mathbb{N}$.



For example, to prove that all natural numbers satisfy a given equation, it is sufficient to prove it for $n = 1$ and then to prove that if the equation hold for n, it holds also for $n + 1$.

☛ ***Example 1.***

Prove that the following equation holds for each $n \in \mathbb{N}$:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\,.$$

Solution. Denote $U = \{n \in N \mid$ (1) holds for $n\}$.

Step 1: $1 \in U$: $\quad 1^2 = \dfrac{1 \cdot 2 \cdot 3}{6}$

Step 2: $n \in U \Rightarrow (n+1) \in U$:

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 =$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}\,,$$

which is the equation (1) for $n = n+1$. $\square$

The set of integers

$$\mathbb{Z} = \{\dots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \dots\}$$

The set of integers is an extension of $\mathbb{N}_0$ containing all solutions of equations of the form

$$n_1 + x = n_2\,, \quad \text{where} \quad n_1,\ n_2 \in \mathbb{N}\,.$$

The set of rational numbers

$$\mathbb{Q} = \left\{ \frac{z}{n},\ \text{where } z \in \mathbb{Z}\,,\ n \in \mathbb{N} \right\}$$

The set of rational numbers is an extension of the set $\mathbb{Z}$ containing the set of all solutions of equations of the form

$$z_1 x = z_2\,, \quad \text{kde} \quad z_1,\ z_2 \in \mathbb{Z}\,.$$

The set of real numbers

$$\mathbb{R} = \{a_0.a_1a_2\ldots a_n\ldots,\ a_0 \in \mathbb{Z}\,, a_k \in \{0,\ \ldots,\ 9\} \text{ pro } k \geq 1\}\ ,$$

where to each $n_0 \in \mathbb{N}$ exists $n > n_0$ such that $a_n \neq 9$.

Without the last condition: the expansion would not be unique, e.g., number 1 would have two expansions:

$$1.000\ldots \qquad \text{a} \qquad 0.999\ldots\ ,$$

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots + \frac{9}{10^n} + \cdots = \frac{9}{10} \cdot \frac{1}{1 - 1/10} = 1\,.$$

Rational numbers: only **finite or periodic decimal expansion**

Ordering or real numbers

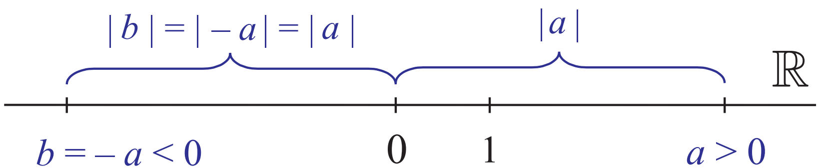
- For any two real numbers a, b, one and only one relation holds: $a < b,\qquad a = b,\qquad a > b.$

- For any three real numbers $a,\ b,\ c,$
  - If $a < b$ and $b < c$, then $a < c$.
  - If $a < b$, then $a + c < b + c$.
  - If $a < b$ and $c > 0$, then $ac < bc$.

- For any four real numbers $a, b, c, d,$
  - If $a < b$ and $c < 0$, then $ac > bc$.
  - If $a < b$ and $c < d$, then $a + c > b + d$.
  - If $0 < a < b$ and $0 < c < d$, then $ac < bd$.
  - Je-li $0 < a < b$, then $1/a < 1/b$.

Absolute value of a real number a :

$$|\,a\,| = \begin{cases} a, & \text{je-li} \quad a \geq 0, \\ -a, & \text{je-li} \quad a < 0. \end{cases}$$



☛ ***Example***

$|5| = 5, \ |-5| = 5 = -(-5)$

Distance of real numbers

For any $x, y, z \in \mathbb{R}$, the following conditions are satisfied:

1. $|x - y| \geq 0$,
2. $|x - y| = 0 \iff x = y$,
3. $|x - z| \leq |x - y| + |y - z|$ (triangle inequality).

On a real axis, $|\, a \,|$ represents **a distance of a from the origin**; $|\, a - b \,|$ represents **a distance of a and b.**

Extended set of real numbers: $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$

New symbols $+\infty, -\infty$ with the property:

$$-\infty < x < +\infty \text{ for any } x \in \mathbb{R}$$

Extension of algebraic operations:

$$|\pm\infty| = +\infty\,, \qquad \pm\infty + x = \pm\infty \text{ for any } x \in \mathbb{R}\,,$$

$$+\infty + (+\infty) = +\infty\,, \qquad -\infty - (-\infty) = -\infty\,,$$

$$x \cdot (\pm\infty) = \pm\infty \text{ for any } x > 0\,,$$

$$x \cdot (\pm\infty) = \mp\infty \text{ for any } x < 0\,,$$

$$\frac{\pm\infty}{x} = \pm\infty \text{ for any } x > 0\,, \qquad \frac{\pm\infty}{x} = \mp\infty \text{ for any } x < 0\,,$$

$$\frac{x}{\pm\infty} = 0 \text{ for any } x \in \mathbb{R}\,.$$

Cannot be defined: $+\infty + (-\infty), \quad 0 \cdot (\pm\infty), \quad 0/0, \quad \pm\infty/\pm\infty\,.$

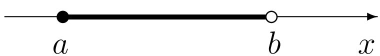
Intervals

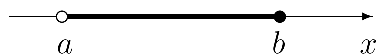
➥ **Bounded interval** (on number axis: piece of a straight line)

➥ **Closed:** $\langle a, b\rangle = \{x \in \mathbb{R}; a \leq x \leq b\}$

➥ **Half-closed:**

$$\langle a, b) = \{x \in \mathbb{R}; a \leq x < b\}\,, \ (a, b\rangle = \{x \in \mathbb{R}; a < x \leq b\}$$

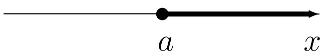


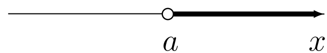


➥ **Open:** $(a, b) = \{x \in \mathbb{R}; a < x < b\}$

- **Unbounded**
- **Left-bounded:**

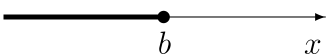
$$\langle a, +\infty) = \{x \in \mathbb{R}; x \geq a\}\,, \;\; (a, +\infty) = \{x \in \mathbb{R}; x > a\}$$

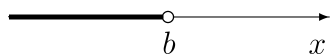




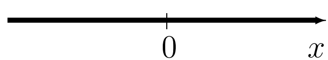
- **Right-bounded:**

$$(-\infty, b\rangle = \{x \in \mathbb{R}; x \leq b\}\,, \;\; (-\infty, b) = \{x \in \mathbb{R}; x < b\}$$



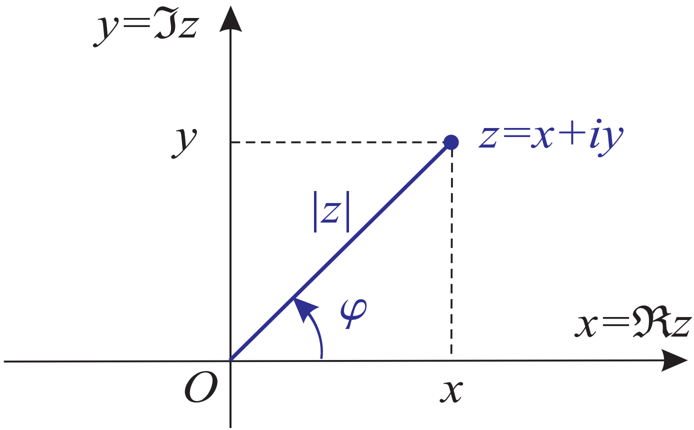


- **Unbounded:** $(-\infty, +\infty) = \mathbb{R}$



The set of complex numbers

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R},\ i^2 = -1\}$$



Absolute value of a complex number $z = x + iy$:

$$|z| = \sqrt{z\bar{z}} \equiv \sqrt{x^2 + y^2}$$

Algebraic form:

$$z = x + iy$$

Goniometric form:

$$z = |z|(\cos\varphi + i\sin\varphi) = |z|\cos\varphi + i|z|\sin\varphi\,.$$

Exponential form:

$$z = |z|e^{i\varphi}\,, \text{ where } e^{i\varphi} = \cos\varphi + i\sin\varphi\,, \quad \varphi \in \mathbb{R}\,.$$

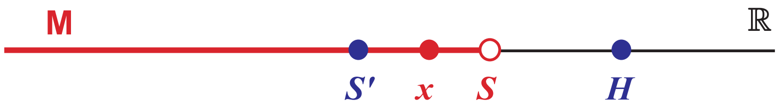
Sets of real numbers and their properties

Definition 1. A set $M \subset \mathbb{R}$ is called

- **bounded from above** if there exists a real number $U \in \mathbb{R}$, called **upper bound**, such that $x \leq U$ for every $x \in M$;
- **bounded from below** if there exists a real number $L \in \mathbb{R}$, called **upper bound**, such that $x \geq L$ for every $x \in M$;
- **bounded** if it is bounded both from above and below.

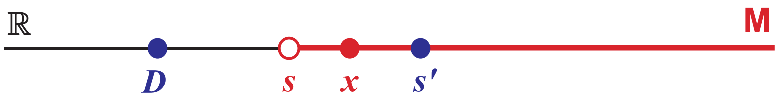
Definition 2. Let $M \subset \mathbb{R}$. A number $S \in \mathbb{R}$ is called **supremum of a set** M if it is the least upper bound, i.e.,

1. for every $x \in M$, $x \leq S$,
2. for every $S' < S$ there exists $x \in M$ such that $x > S'$.



Definition 3. Let $M \subset \mathbb{R}$. A number $s \in \mathbb{R}$ is called **infimum of a set** M if it is the greatest lower bound, i.e.,

1. for every $x \in M$, $x \geq s$,
2. for every $s' > s$ there exists $x \in M$ such that $x < s'$.



Theorem 1 on supremum and infimum in $\mathbb{R}$

Every nonempty set of real numbers that is bounded from above has a supremum, and every nonempty set of real numbers that is bounded from below has an infimum.

Remark. Notice that the proposition does not hold in the set of rational numbers. It is sufficient to consider

$$M = \left\{ q \in \mathbb{Q} ;\ q^2 < 2 \right\} .$$

In $\mathbb{R}$:
supremum: $\sqrt{2}$, infimum: $-\sqrt{2}$ (not rational!)

In $\mathbb{Q}$:
supremum and infimum do not exist

Definition 4. Let $\varepsilon > 0$. **An** $\varepsilon-$**neighbourhood of** $a \in \mathbb{R}$:

$$U_\varepsilon(a) = \left\{x \in \mathbb{R};\ |x-a| < \varepsilon\right\}.$$

Definition 5. A punctured $\varepsilon-$**neighbourhood (also ring neighbourhood) of** $a \in \mathbb{R}$: $P_\varepsilon(a) = U_\varepsilon(a) \setminus \{a\}$, i.e.,

$$P_\varepsilon(a) = \left\{x \in \mathbb{R};\ 0 < |x-a| < \varepsilon\right\}.$$

Let $\varepsilon > 0$. **An $\varepsilon-$neighbourhood of $+\infty$** is defined as

$$U_\varepsilon(+\infty) = \left\{x \in \mathbb{R}\,;\; x > \frac{1}{\varepsilon}\right\} \cup \{+\infty\},$$

and **an $\varepsilon-$neighbourhood of $-\infty$** is defined as

$$U_\varepsilon(-\infty) = \left\{x \in \mathbb{R}\,;\; x < -\frac{1}{\varepsilon}\right\} \cup \{-\infty\}\,.$$

Punctured $\varepsilon-$neighbourhoods of $\pm\infty$:

$$P_\varepsilon(+\infty) = \left\{x \in \mathbb{R}\,;\; x > \frac{1}{\varepsilon}\right\}\,,$$

$$P_\varepsilon(-\infty) = \left\{x \in \mathbb{R}\,;\; x < -\frac{1}{\varepsilon}\right\}\,.$$

Set $\mathbb{R}^n$ and its subsets

In the second half of this semester, we will work with the set of ordered n-tuples of real numbers, usually denoted as $\mathbb{R}^n$:

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n);\ x_1, x_2, \ldots, x_n \in \mathbb{R}\} \tag{1.1}$$

i.e.,

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$$

Multiplication by a real number a and addition in $\mathbb{R}^n$:

$$a\boldsymbol{x} = a\big(x_1, x_2, \ldots, x_n\big) = \big(ax_1, ax_2, \ldots, ax_n\big)\,,$$

$$\boldsymbol{x}+\boldsymbol{y} = \big(x_1, x_2, \ldots, x_n\big)+\big(y_1, y_2, \ldots, y_n\big) = \big(x_1+y_1, x_2+y_2, \ldots, x_n+y_n\big)\,.$$

As we know from linear algebra: $\mathbb{R}^n$ with these operations is a vector space over a field of real numbers.

Distance of two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{y} - \boldsymbol{x}\| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}. \tag{1.2}$$

Definition 6. Let $M \in \mathbb{R}^n$, $\varepsilon > 0$. **An** $\varepsilon-$**neighbourhood of** a is a set $U_\varepsilon(\boldsymbol{a}) = \{\boldsymbol{x} \in M \mid d(\boldsymbol{x}, \boldsymbol{a}) < \varepsilon\}$.

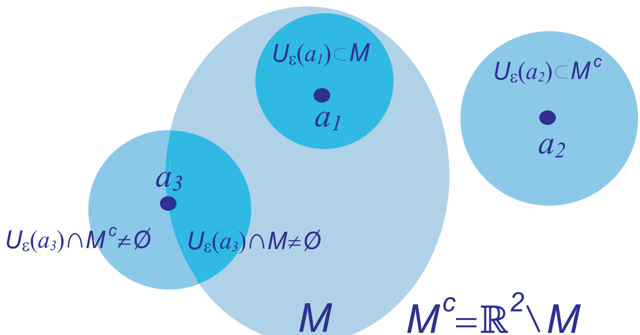
A ε**-punctured neighbourhood of** a is a set

$$P_\varepsilon(\boldsymbol{a}) = \{\boldsymbol{x} \in M \mid 0 < d(\boldsymbol{x}, \boldsymbol{a}) < \varepsilon\}.$$

Remark. Obviously: $P_\varepsilon(\boldsymbol{a}) = U_\varepsilon(\boldsymbol{a}) \setminus \{\boldsymbol{a}\}$.

Definition 7. Consider a set $M \subset \mathbb{R}^n$. A point $\boldsymbol{a}$ is called

- **an interior point of** M if there exists $U_\varepsilon(\boldsymbol{a}) \subset M$.
- **an exterior point of** M it there exists $U_\varepsilon(\boldsymbol{a})$ such that $U_\varepsilon(\boldsymbol{a}) \cap M = \emptyset$.
- **a boundary point of** M if every $U_\varepsilon(\boldsymbol{a})$ has a non-empty intersection both with M and $M^C = \mathbb{R} \setminus M$.

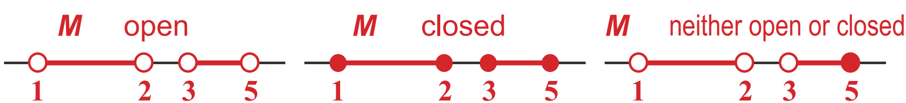


Definition 8. A set $M \subset \mathbb{R}^n$ is called

- **open** if and only if all its points are interior,
- **closed** if and only if its complement $M^C = \mathbb{R}^n \setminus M$ is an open set.

Example 2.

- $M = (-1, 2) \cup (3, 5)$ is an open set.
- $M = \langle -1, 2\rangle \cup \langle 3, 5\rangle$ is a closed set.
- $M = (-1, 2) \cup (3, 5\rangle$ is neither open or closed.



Definition 9. The set of all interior points of a set M is denoted by M° and called **interior of** M.

Definition 10. A complement to an interior of M is called **closure of** M. It is denoted by $\overline{M}$, i.e., $\overline{M} = \mathbb{R} \setminus \big(\mathbb{R} \setminus M\big)^\circ$.

Definition 11. The set of all boundary points of M is denoted by ∂M and called **boundary of** M.

☛ ***Example 3.***

Consider $M = (-1, 2) \cup (3, 5\rangle$.

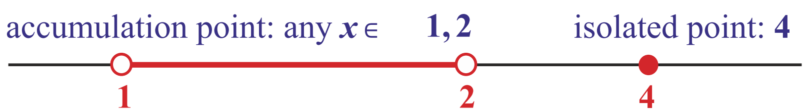
$$M^\circ = (-1,2)\cup(3,5)\ , \quad \overline{M} = \langle -1,2\rangle\cup\langle 3,5\rangle\ , \quad \partial M = \{-1,2,3,5\}\,.$$

Definition 12. A point x is called

- **an accumulation point** of a set M if $M \cap P_\varepsilon(\boldsymbol{x}) \neq \emptyset$ for every punctured neighbourhood $P_\varepsilon(\boldsymbol{x})$,
- **isolated point** of a set M if there exists a punctured neighbourhood $P_\varepsilon(\boldsymbol{x})$ such that $M \cap P_\varepsilon(\boldsymbol{x}) = \emptyset$.

Example 4.

$M \quad (1,2) \cup \{4\}$



Definition 13. $M \subset \mathbb{R}^n$ is called **bounded** if there exists $K \in \mathbb{R}$ such that

$$d(\boldsymbol{x}, \boldsymbol{O}) \leq K.$$

for every $\boldsymbol{x} \in M$.

Definition 14. A closed and bounded set $M \subset \mathbb{R}^n$ is called **compact**.