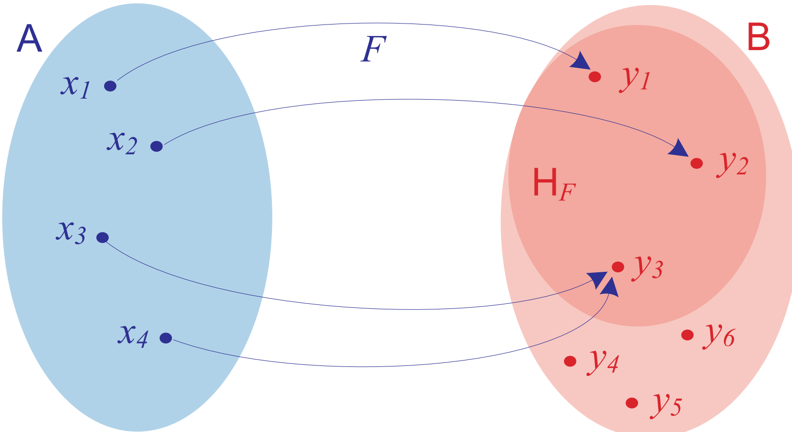
CHAPTER 3

FUNCTIONS

Mapping and Function



Consider two non-empty sets A, B. As we already know, **a mapping of a set A to B** is defined as a set F of ordered pairs $(x, y) \in$ **A** $\times$ **B**, where for every $x \in$ **A** there exists **exactly one** element $y \in$ **B** such that $(x, y) \in F$.

An element x is called a **preimage** of an element y, an element y is called an **image** of x in the mapping F. We also say that y is the **value** of the mapping F in a point x and write $y = F(x)$ or $x \mapsto F(x)$. A set **A** is called a **domain of a mapping** F and it is also denoted by a symbol $\mathbf{D}(F)$ or $\mathbf{D}_F$. The set of all images in the mapping F is called **range of the mapping** F and it is denoted by $\mathbf{H}(F)$ or $\mathbf{H}_F$. It is $\mathbf{H}(F) \subset \mathbf{B}$.

Symbolically, a mapping F from **A** to **B** is expressed as follows:

$$F : \mathbf{A} \to \mathbf{B}, \quad \mathbf{D}(F) = \mathbf{A}$$

Special cases of a mapping F of a set A to a set B

- **A mapping in a set A** or **a mapping of a set A to itself** is a mapping F where **A** $=$ **B**.

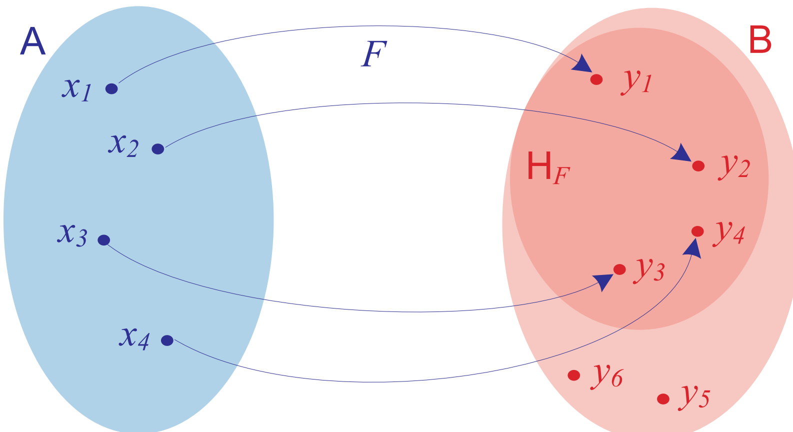
  For example:

  - **A real function of one real variable** is a mapping in a set of all real numbers **R**, i.e.,

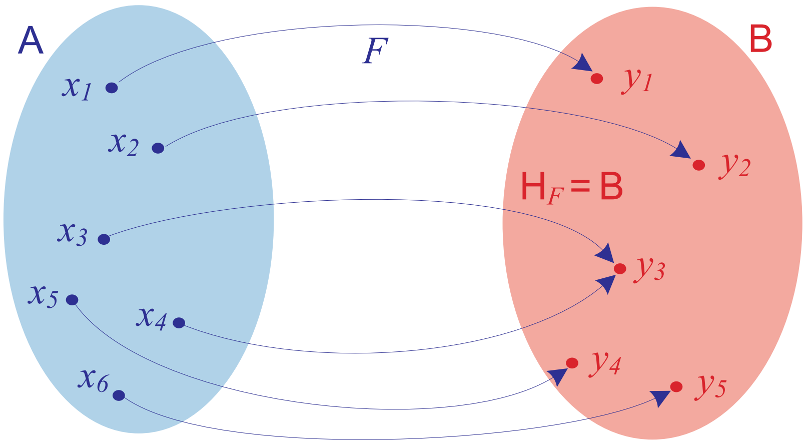
    $$\mathbf{A} = \mathbf{B} = \mathbb{R}.$$

  - **Geometric mappings in plane and space**, where **A**, **B** are sets of points in the same plane or in space.

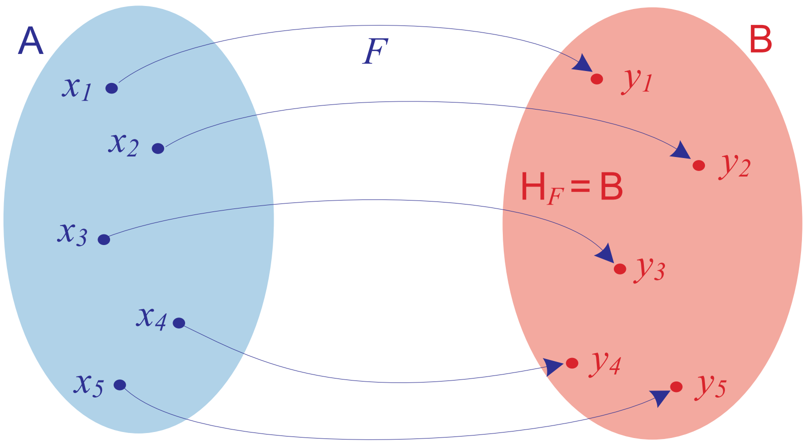
➥ **Invertible** or **One-to-one mapping** is a mapping F such that every element $y \in \mathbf{H}(F)$ is an image of **exactly one** element $x \in \mathbf{A} = \mathbf{D}(F)$, i.e., **any two different preimages** x_1, x_2 **have also different images** $F(x_1), F(x_2)$.



➥ **A mapping of a set A onto a set B** is a mapping F such that every element of **B** is an image of at least one element of a set **A**, tj. $\mathbf{B} = \mathbf{H}(F)$.

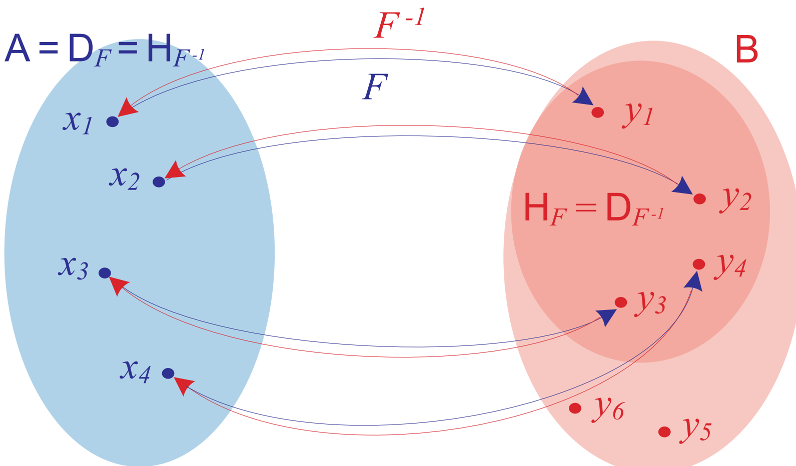


➥ **A bijection of a set A to B** a **one-to-one mapping of a set A onto a set B**.

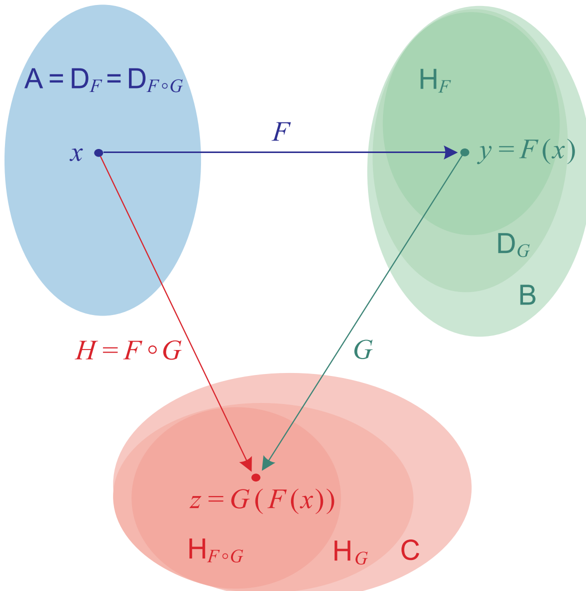


If a given mapping F is **one-to-one**, then there exists exactly one one-to-one mapping which assigns a preimage $x \in \mathbf{D}(F)$ to every element $y \in \mathbf{H}(F)$. This mapping is called **inverse mapping to F** and it is usually denoted by a symbol F^{-1}. Obviously: $\mathbf{D}(F^{-1}) = \mathbf{H}(F)$, $\mathbf{H}(F^{-1}) = \mathbf{D}(F)$,

$$x = F^{-1}(y) \text{ if and only if } y = F(x)$$



Let G and F be two mappings such that $\mathbf{H}_F \subset \mathbf{D}_G$. A mapping H is called a **composition of mappings F and G**, if $H(x) = G(F(x))$ for all $x \in \mathbf{D}_F$. A composition of mappings F and G (in this order) is denoted as $H = F \circ G$.

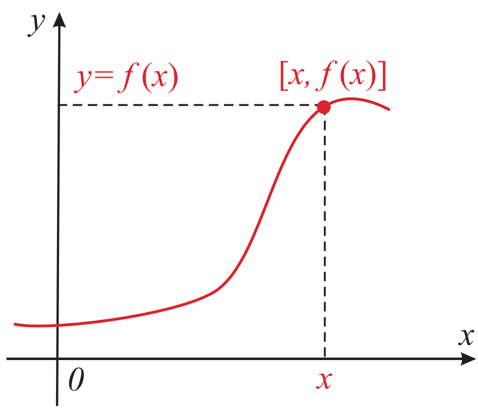


Real functions of one real variable

Real function of one real variable f is a mapping in the set of real numbers **R**; a preimage x is called **variable** or **argument of a function** f, an image $y = f(x)$ is called **function value.**

Graph of a function f is a set of all points $(x, f(x))$ in a plane with a given cartesian system of coordinates:

$$\text{graf } f = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbf{D}(f),\ y = f(x)\}$$



Properties and types of functions

➥ Even and odd functions

Let f be a function such that $-x \in \mathbf{D}(f)$ for all $x \in \mathbf{D}(f)$.

- ➥ f is called an **even function** if

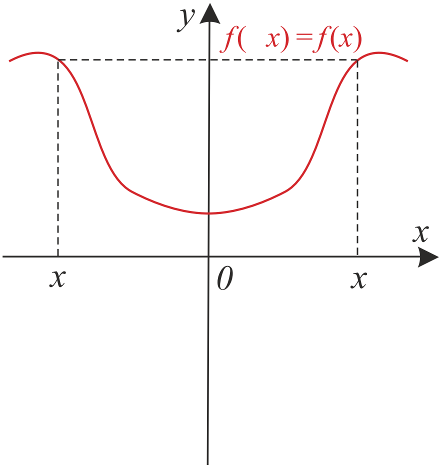
$$f(-x) = f(x) \quad \text{for all} \quad x \in \mathbf{D}(f).$$

- ➥ f is called an **odd function** if

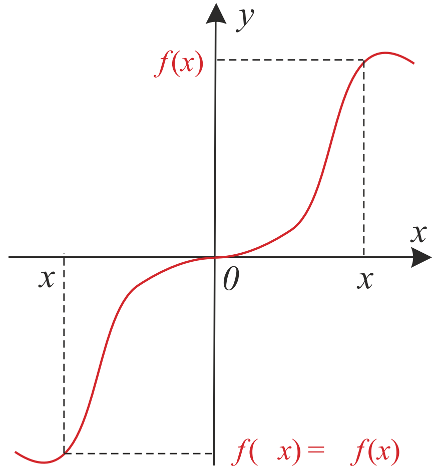
$$f(-x) = -f(x) \quad \text{for all} \quad x \in \mathbf{D}(f).$$

(Of course, many functions are NEITHER even nor odd.)

Even function:

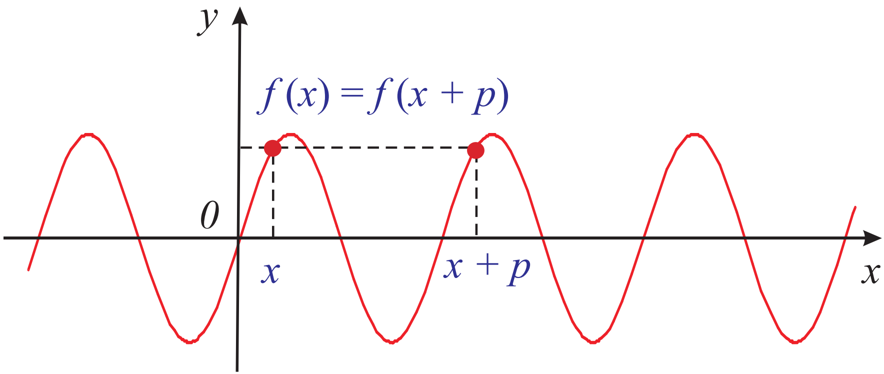


Odd function:



➥ Periodic functions

A function f is called **periodic** if there exists a real number $p \neq 0$ such that $x \pm p \in \mathbf{D}(f)$ and $f(x \pm p) = f(x)$ for all $x \in \mathbf{D}(f)$.

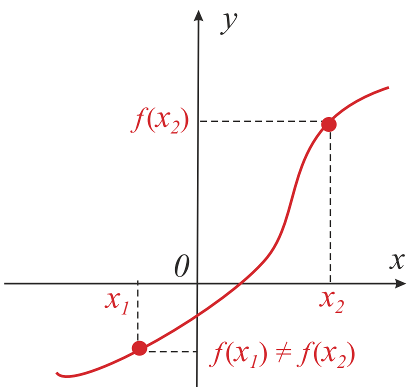


➥ One-to-one functions and their inverses

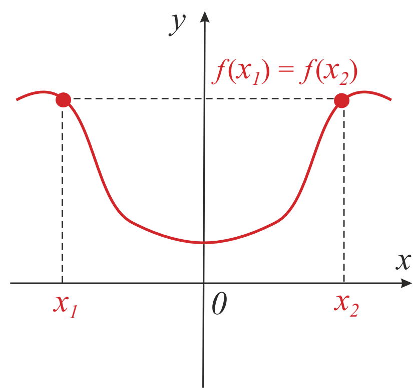
A function is a special case of a mapping, the definitions are therefore the same as for mappings:

A function f is called **one-to-one** or **invertible**, if

$$f(x_1) \neq f(x_2) \quad \text{for all} \quad x_1\,, x_2 \in \mathbf{D}_f\,,\ x_1 \neq x_2.$$



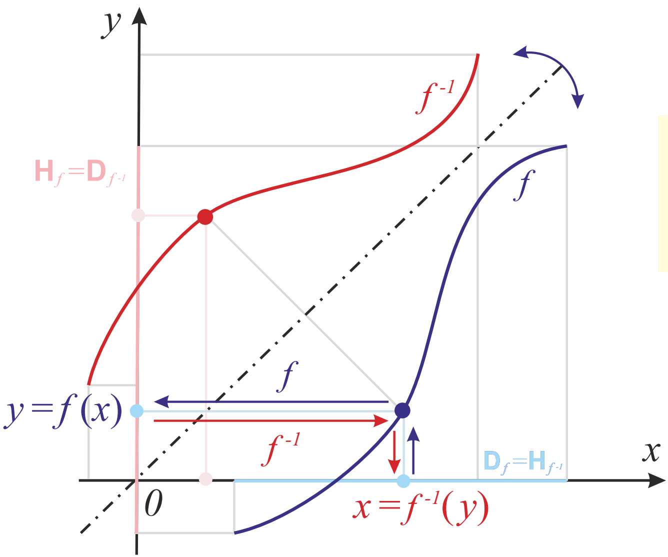
One-to-one function



This function is not one-to-one

If a function f is one-to-one, then there exists its **inverse function** f^{-1}, which assigns to every $y \in \mathbf{H}_f$ its preimage $x \in \mathbf{D}_f$:

$$x = f^{-1}(y) \text{ if and only if } y = f(x).$$

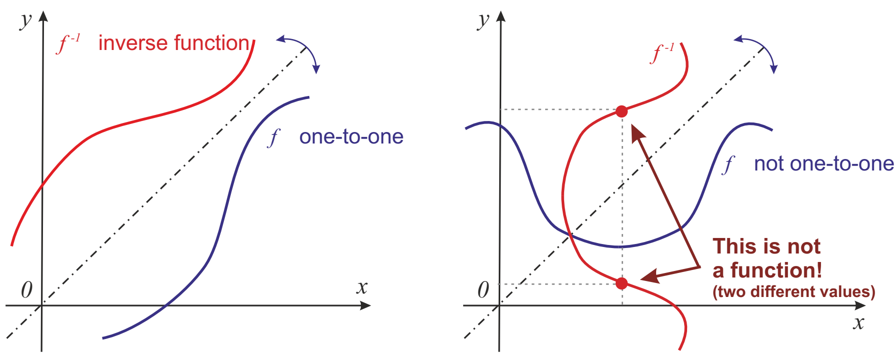


$\mathbf{D}_f = \mathbf{H}_{f^{-1}}$

$\mathbf{H}_f = \mathbf{D}_{f^{-1}}$

Construction of a graph of an inverse function: variable on an axis x and values of an inverse function on y – compared to the graph of f, the coordinate axis have "changed their roles", i.e., the graph of f^{-1} is symmetrical to the graph of f in an axial symmetry with respect to the axis of the first and third quadrant.

Notice that an inverse function exists only for a one-to-one function. If a function is not one-to-one, then the resulting curve is not a function:

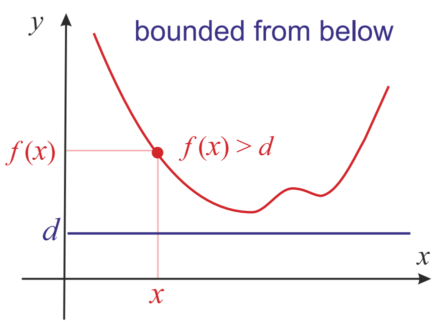


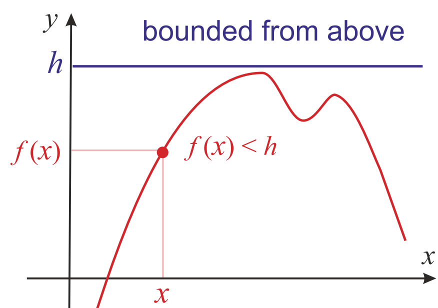
➥ Functions bounded from below, from above or bounded

Consider a function f and a subset **M** of its domain **D**(f).

- ➥ f is called **bounded from below on the set M** if there exists $d \in$ **R** such that $f(x) \geq d$ for all $x \in$ **M**.
- ➥ f is called **bounded from above on the set M** if there exists $h \in$ **R** such that $f(x) \leq h$ for all $x \in$ **M**.
- ➥ f is called **bounded on the set M** if it is bounded both from below and above on **M**.

  If **M** $= D_f$, we say that f **bounded (from below, from above).**



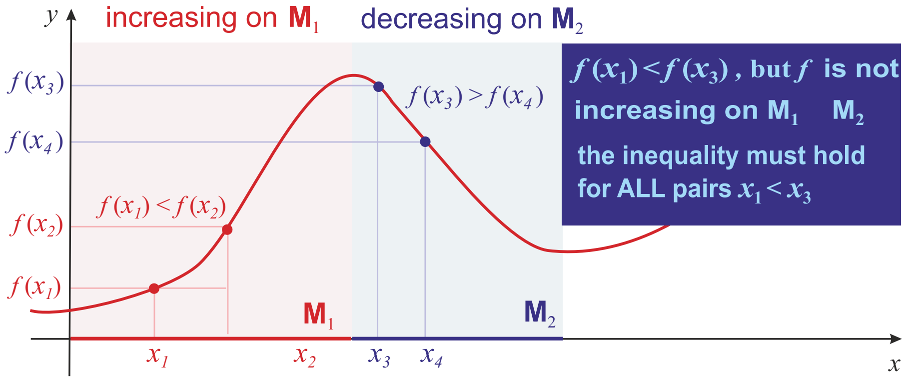


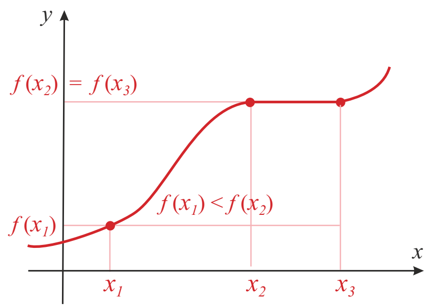
➥ Monotonic (monotone) functions

Consider a function f and a subset $\mathbf{M} \subset \mathbf{D}(f)$.

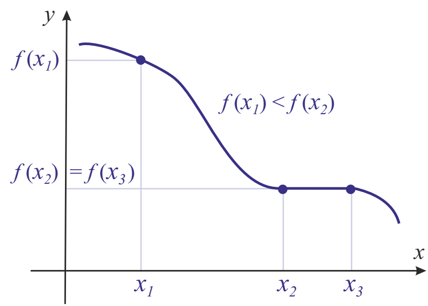
- ➥ f is called **increasing on the set M** if
  $f(x_1) < f(x_2)$ for all $x_1, x_2 \in \mathbf{M}$, $x_1 < x_2$.
- ➥ f is called **decreasing on the set M** if
  $f(x_1) > f(x_2)$ for all $x_1, x_2 \in \mathbf{M}$, $x_1 < x_2$.
- ➥ f is called **non-decreasing on the set M** if
  $f(x_1) \leq< f(x_2)$ for all $x_1, x_2 \in \mathbf{M}$, $x_1 < x_2$.
- ➥ f is called **non-increasing on the set M** if
  $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in \mathbf{M}$, $x_1 < x_2$.

Functions that are either increasing or decreasing are called **strictly monotonic** (on the given set); non-decreasing and non-increasing functions are called **monotonic** (on the given set).





non-increasing



non-decreasing

Definition 1. We say that a function f has a

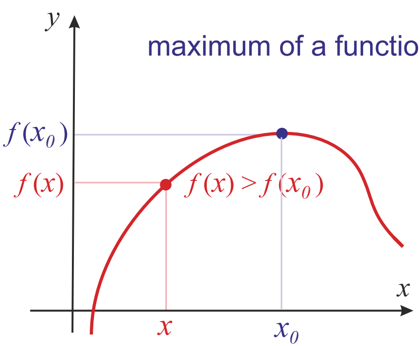
- **maximum in** $x_0 \in \mathbf{D}(f)$ if

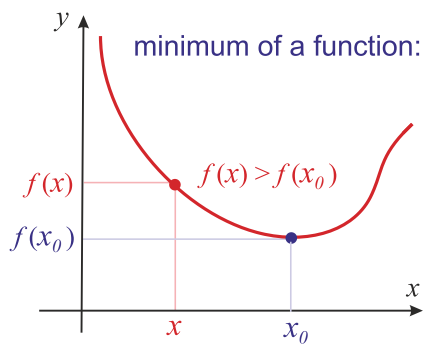
$$f(x_0) \geq f(x) \quad \text{for all} \quad x \in \mathbf{D}(f),$$

- **minimum in** $x_0 \in \mathbf{D}(f)$ if

$$f(x_0) \leq f(x) \quad \text{for all} \quad x \in \mathbf{D}(f).$$

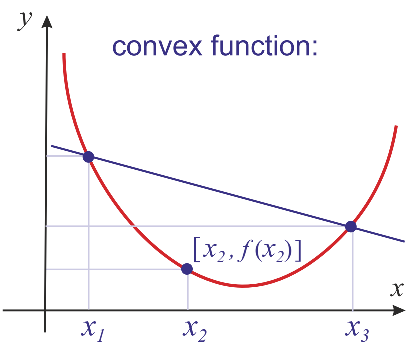
We speak also about **(global) extremes.**

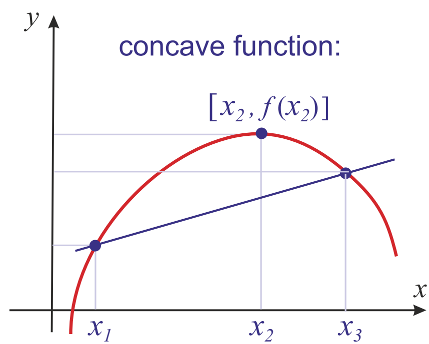




Definition 2. Let $I \subset \mathbb{R}$ be interval, $f : I \to \mathbb{R}$ a function. If for all $x_1, x_2, x_3 \in I$, where $x_1 < x_2 < x_3$, a point $A = [x_2, y]$ of a line passing through the points $\big[x_1; f(x_1)\big]$ and $\big[x_3; f(x_3)\big]$ of the graph of f lies

- above the point $[x_2, f(x)]$, then f is called **convex on the interval** I,
- below the point $[x_2, f(x)]$, then f is called **concave on the interval** I,

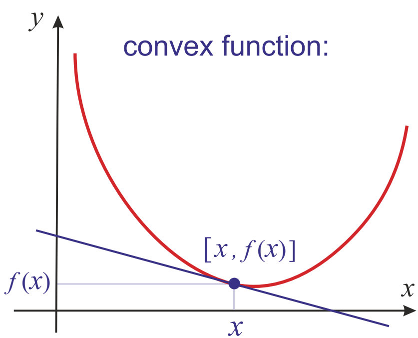


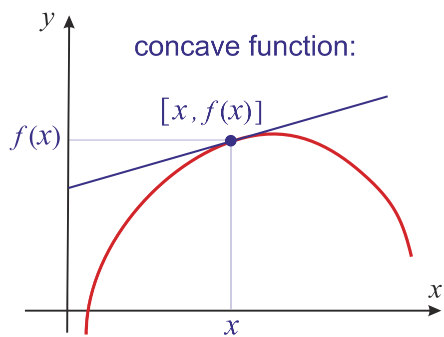


Alternatively, we can say:

Definition 3. Let $I \subset \mathbb{R}$ be interval, $f : I \to \mathbb{R}$ a function. If the graph of f on the interval I lies

- above the tangent in any point $x \in I$, then f is called **convex on the interval** I,
- below the tangent in any point $x \in I$, then f is called **concave on the interval** I,



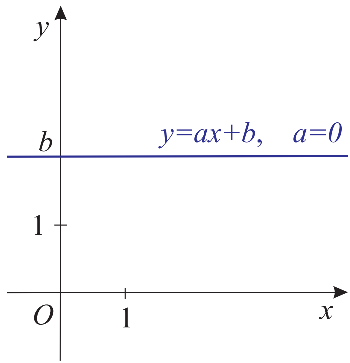


Basic elementary functions

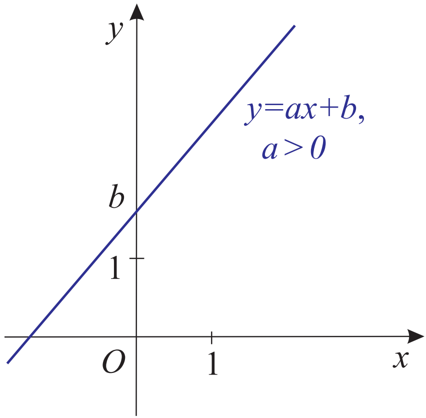
Linear function

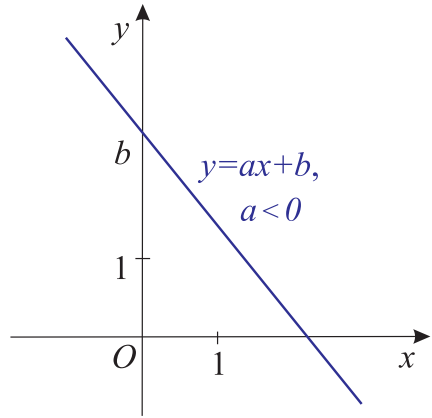
A linear function is any function

$$f: \; y = ax + b, \quad \mathbf{D}(f) = \mathbb{R}.$$



$\mathbf{H}$: $\mathbf{D}(f) = \mathbb{R}$, $\mathbf{H}(f) = \{b\}$, non-increasing and non-decreasing, not one-to-one





$$\mathbf{D}(f) = \mathbb{R},\ \mathbf{H}(f) = \mathbb{R}$$

bounded neither from above nor below

increasing	decreasing
one-to-one	one-to-one

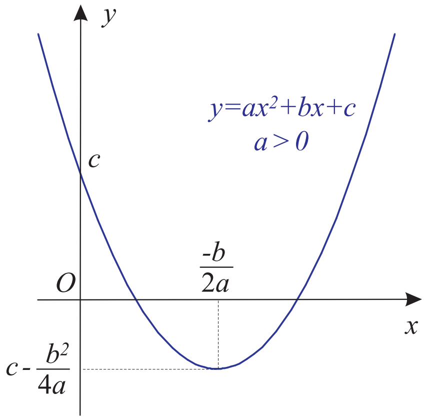
Quadratic function

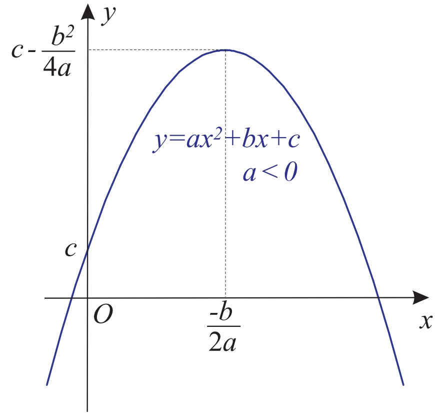
A quadratic function is any function

$$f: \ y = ax^2 + bx + c, \quad a \neq 0, \quad \mathbf{D}(f) = \mathbb{R}.$$

Graph of any quadratic function: a **parabola** symmetric to a vertical axis o.

An intersection of a parabola with its axis of symmetry o is called a **vertex.**





$\mathbf{D}(f) = \mathbb{R},\ \mathbf{H}(f) = \left[c - \dfrac{b^2}{4a}, +\infty\right)$

bounded from below, not from above

decreasing in $\left(-\infty, -\dfrac{b}{2a}\right]$

increasing in $\left[-\dfrac{b}{2a}, +\infty\right)$

$\mathbf{D}(f) = \mathbb{R},\ \mathbf{H}(f) = \left(-\infty, c - \dfrac{b^2}{4a}\right]$

bounded from above, not from below

increasing in $\left(-\infty, -\dfrac{b}{2a}\right]$

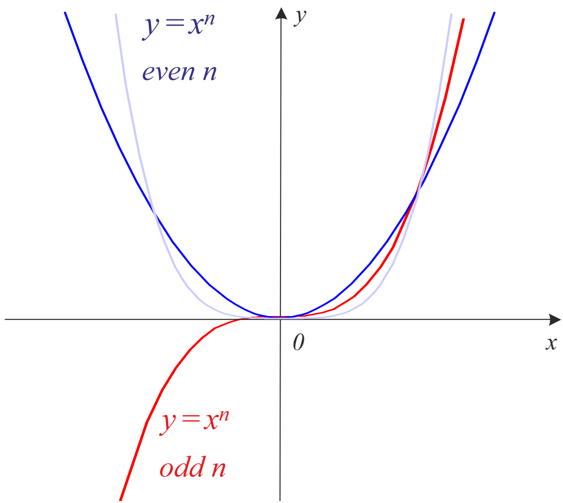
decreasing in $\left[-\dfrac{b}{2a}, +\infty\right)$

Power function with a natural exponent

A power function with a natural exponent is any function

$$f: \ y = x^n, \quad n \in \mathbf{N}, \quad \mathbf{D}(f) = \mathbb{R}.$$

f is linear for $n = 1$, quadratic for $n = 2$. For $n > 1$, its graph is a **parabola of the degree n**.



By algebraic operations of functions $f(x) = x^n$, we get **polynomials**.

Polynomial is any function $P : \mathbb{R} \to \mathbb{R}$ of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ a_i \in \mathbb{R}.$$

If the polynomial is not identically equal to zero, then there exists a maximal n such that $a_n \neq 0$. This n is called **degree** of a polynomial P. In the following, we will suppose that $P(x)$ is not identically equal to zero and it has a degree n.
A root of a polynomial P is apoint $x_0 \in \mathbb{R}$ such that

$$P\big(x_0\big) = 0.$$

If x_1 is a root of $P(x)$ of degree n, we can write

$$P(x) = (x - x_1)P_1(x),$$

where $P_1(x)$ is a polynomial of degree $(n-1)$. Similarly, if x_2 is a root of $P_1(x)$, it is $P_1(x) = (x - x_2)P_2(x)$, and thus

$$P(x) = (x - x_1)(x - x_2)P_2(x),$$

where $P_2(x)$ is a polynomial of the degree $(n-2)$, etc. Any polynomial can be written in the form

$$P(x) = (x-x_1)^{k_1}(x-x_2)^{k_2}\dots(x-x_r)^{k_r}P_N(x),$$

where x_1, ..., x_r are pairwise different roots of $P(x)$. Natural numbers k_i are called *multiplicity of the root* x_i and they satisfy $N = k_1 + k_2 + \dots + k_r$, $P_N(x)$ is a polynomial of degree $(n-N)$ which does not have real roots.
Generally, any polynomial of degree n can be written in the form

$$P(x) = a_n\big(x-x_1\big)^{k_1}\big(x-x_2\big)^{k_2}\dots\big(x-x_r\big)^{k_r}\big(x^2+p_1x+q_1\big)^{m_1}$$
$$\big(x^2+p_2x+q_2\big)^{m_2}\dots\big(x^2+p_sx+q_s\big)^{m_s},$$

where the polynomials $x^2 + p_ix + q_i$ do not have real roots and

$$k_1 + k_2 + \dots + k_r + 2m_1 + \dots + 2m_s = n.$$

Rational functions are functions of the form

$$f(x) = \frac{P(x)}{Q(x)} \ ,$$

where $P(x), Q(x)$ are polynomials.
Denote by X_0 the set of all real roots of $Q(x)$. Then the domain of f is $D_f = \mathbb{R} \setminus X_0$.
If the degree n of $P(x)$ is higher or equal to the degree m of $Q(x)$, the function can be written in the form

$$f(x) = P_1(x) + \frac{R(x)}{Q(x)},$$

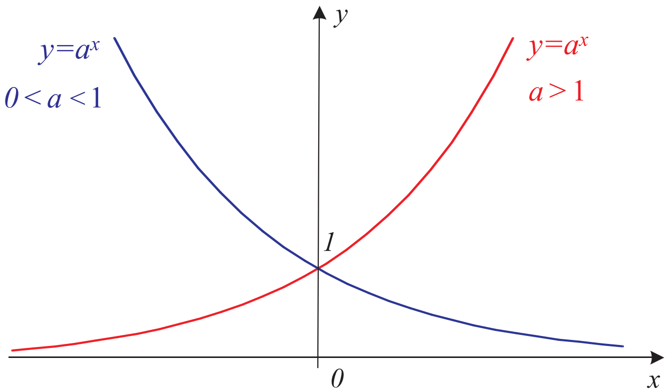
where $P_1(x)$ is a polynomial of degree $(n - m)$ and the degree of the polynomial $R(x)$ is lower than the degree of $Q(x)$.

Exponential function with the basis *a*

Exponential function with the basis *a* is a function

$$f:\ y=a^x,\quad a>0,\quad a\neq 1,\quad D(f)=\mathbb{R}.$$

It is increasing in $\mathbb{R}$ for $a>1$ and decreasing in $\mathbb{R}$ for $0<a<1$. In both cases, it is one-to-one in the whole domain.



Logaritmic function with the basis *a*

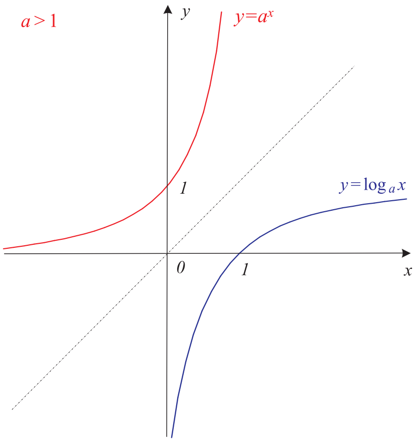
Logaritmic function with the basis *a* is defined as **an inverse function to the exponential function with the same basis** a. Symbolically:

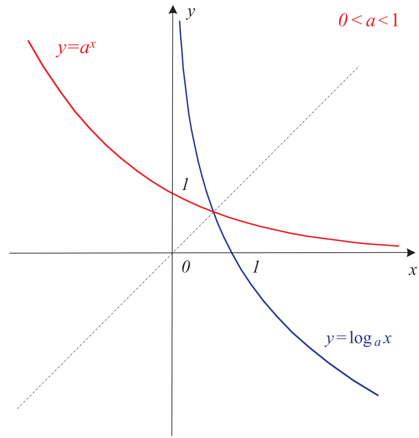
$$f:\ y = \log_a x, \quad a > 0, \quad a \neq 1, \quad D(f) = (0, +\infty).$$

From the definition:

$$y = \log_a x \ \Leftrightarrow \ x = a^y$$

holds for all $x \in (0, +\infty),\ y \in \mathbb{R},\ a > 0,\ a \neq 1.$
The function $\log_a x$ is increasing in $\mathbb{R}$ for $a > 1$ and decreasing in $\mathbb{R}$ for $0 < a < 1$. In both cases, it is one-to-one in the whole domain.


a > 1
y
y=a^x
1
y=log_a x
0
1
x


y
0 < a < 1
y=a^x
1
0
1
x
y=log_a x

Some important formulas:

For $a > 0$, $a \neq 1$, $x, y > 0$ and $r \in \mathbb{R}$, the following equations hold:

$$\log_a(x \cdot y) = \log_a x + \log_a y$$

$$\log_a(x/y) = \log_a x - \log_a y$$

$$\log_a(x^r) = r \cdot \log_a x$$

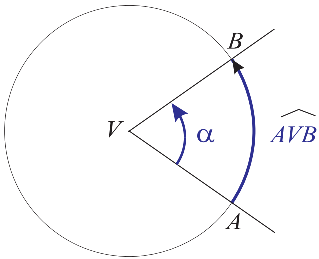
Different basis:

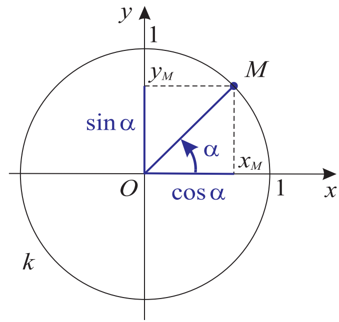
For $a, b > 0$, $a \neq 1$:

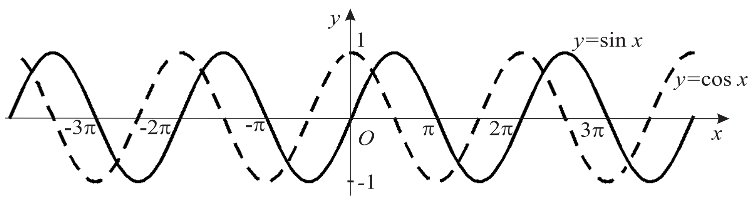
$$\log_a b = \frac{\ln b}{\ln a}.$$

Goniometric functions

$f: \ y = \sin x, \quad \mathbf{D}(f) = \mathbb{R}, \qquad\qquad f: \ y = \cos x, \quad \mathbf{D}(f) = \mathbb{R}.$







The function sinus is odd, cosinus is even, both functions are periodic with the period 2π. Both are bounded:

$$-1 \leq \sin x \leq 1, \qquad -1 \leq \cos x \leq 1.$$

For all $x \in \mathbb{R}$, it is:

$$\sin^2 x + \cos^2 x = 1.$$

$\sin x = 0$ if and only if $x = k\pi = 2k \cdot \frac{\pi}{2}$, where $k \in \mathbb{Z}$

$\cos x = 0$ if and only if $x = (2k+1)\frac{\pi}{2}$, where $k \in \mathbb{Z}$

$$f: \; y = \tan x = \frac{\sin x}{\cos x}, \quad \mathbf{D}(f) = \mathbb{R} - \bigcup_{k \in \mathbb{Z}} \left\{(2k+1)\tfrac{\pi}{2}\right\}$$

$$f: \; y = \cot x = \frac{\cos x}{\sin x}, \quad \mathbf{D}(f) = \mathbb{R} - \bigcup_{k \in \mathbb{Z}} \{k\pi\}.$$

Important values of goniometric functions:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	not def.	0	not def.	0
$\cot\alpha$	not def.	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	not def.	0	not def.

Some relations

Addition formulas

$$\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$$

$$\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \cdot \tan y}$$

$$\cot g\,(x \pm y) = \frac{\pm \cot x \cdot \cot y - 1}{\cot x \mp \cot y}$$

Formulas for a double angle

$$\sin 2x = 2\sin x \cdot \cos x \qquad \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

Formulas for a half-angle

$$\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \qquad \tan \frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$$

$$\cos \frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

The sign depends on the quadrant.

Further addition formulas

$$\sin x + \sin y = 2 \cdot \sin \tfrac{x+y}{2} \cdot \cos \tfrac{x-y}{2}$$

$$\sin x - \sin y = 2 \cdot \cos \tfrac{x+y}{2} \cdot \sin \tfrac{x-y}{2}$$

$$\cos x + \cos y = 2 \cdot \cos \tfrac{x+y}{2} \cdot \cos \tfrac{x-y}{2}$$

$$\cos x - \cos y = -2 \cdot \sin \tfrac{x+y}{2} \cdot \sin \tfrac{x-y}{2}$$

Odd multiples

$$\sin\left(\tfrac{\pi}{2} - \alpha\right) = \cos\alpha \qquad \tan\left(\tfrac{\pi}{2} - \alpha\right) = \cot\alpha$$

$$\cos\left(\tfrac{\pi}{2} - \alpha\right) = \sin\alpha \qquad \cot\left(\tfrac{\pi}{2} - \alpha\right) = \tan\alpha$$

Cyklometric functions

Cyklometric functions are introduced as inverse functions to goniometric functions restricted to an interval on which they are one-to-one.

Arcus sinus,

$$f:\ y = \arcsin x, \quad \mathbf{D}(f) = [-1, 1]\,,$$

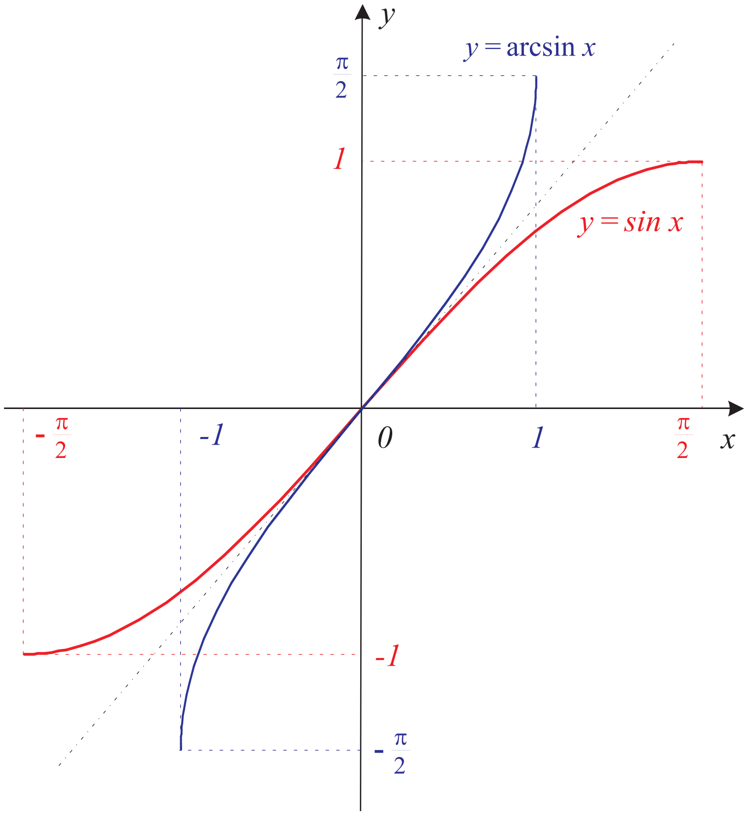
is defined as an inverse function to the function $\sin x$ on the interval $[-\pi/2, \pi/2]$. Je tedy ur$v$cena vztahem

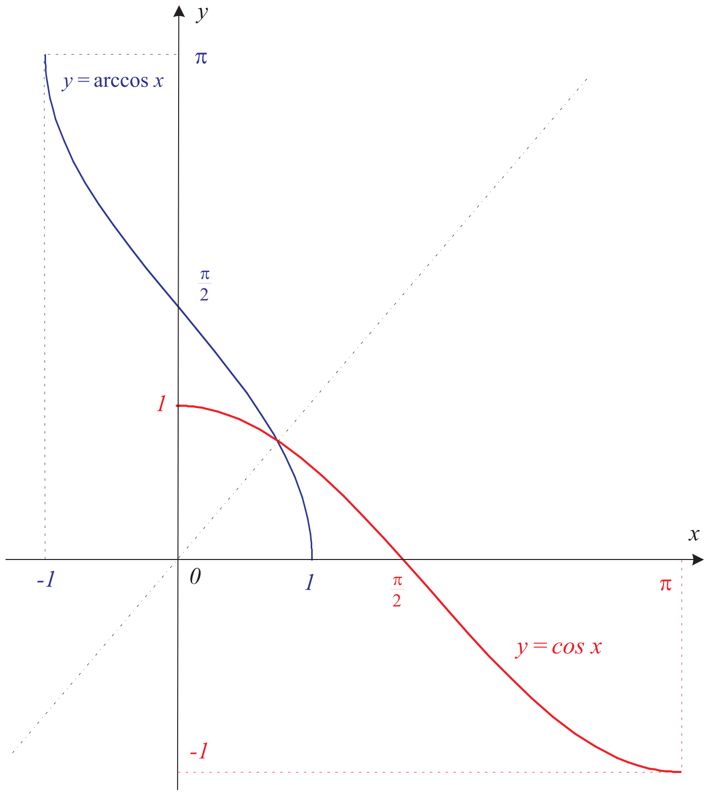
$$y = \arcsin x \iff x = \sin y, \quad y \in [-\pi/2, \pi/2]\,.$$

Funkce arkuscosinus,

$$f:\ y = \arccos x, \quad \mathbf{D}(f) = [-1, 1]\,,$$

is defined as an inverse function to $\cos x$ on the interval $[0, \pi]$, i.e.:
$y = \arccos x \iff x = \cos y, \quad y \in [0, \pi]\,.$


y
y = arcsin x
π/2
1
y = sin x
- π/2
-1
0
1
π/2
x
-1
- π/2


y
π
y = arccos x
π/2
1
x
-1
0
1
π/2
π
y = cos x
-1

Arcus tangent,

$$f: \; y = \operatorname{arctg} x, \quad \mathbf{D}(f) = \mathbb{R}\,,$$

is defined as an inverse function to $\tan x$ on the interval $(-\pi/2, \pi/2)$, i.e.,

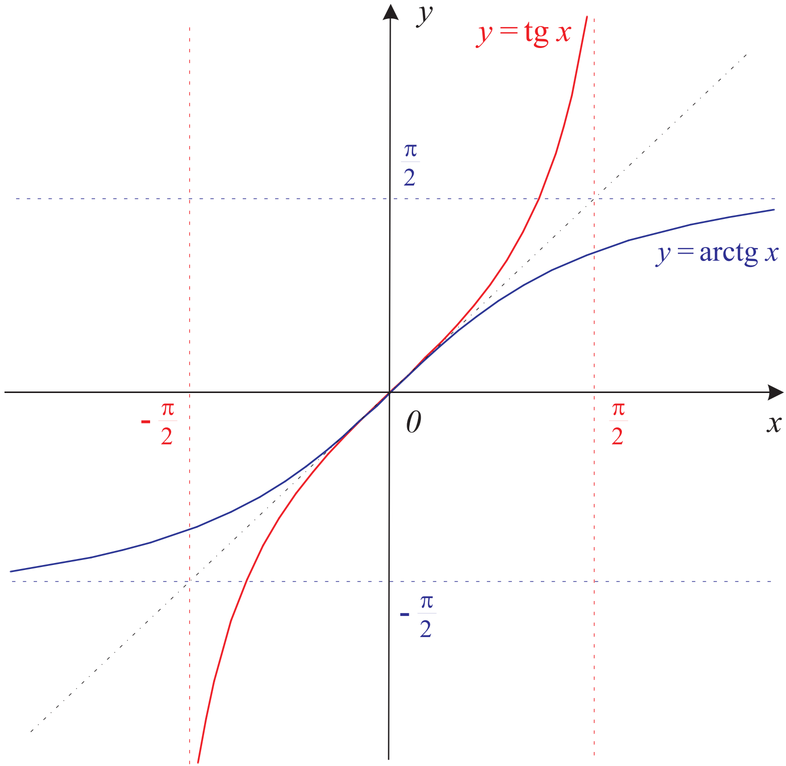
$$y = \operatorname{arctg} x \quad \Longleftrightarrow \quad x = \tan y, \quad y \in (-\pi/2, \pi/2).$$

Arcus cotangent,

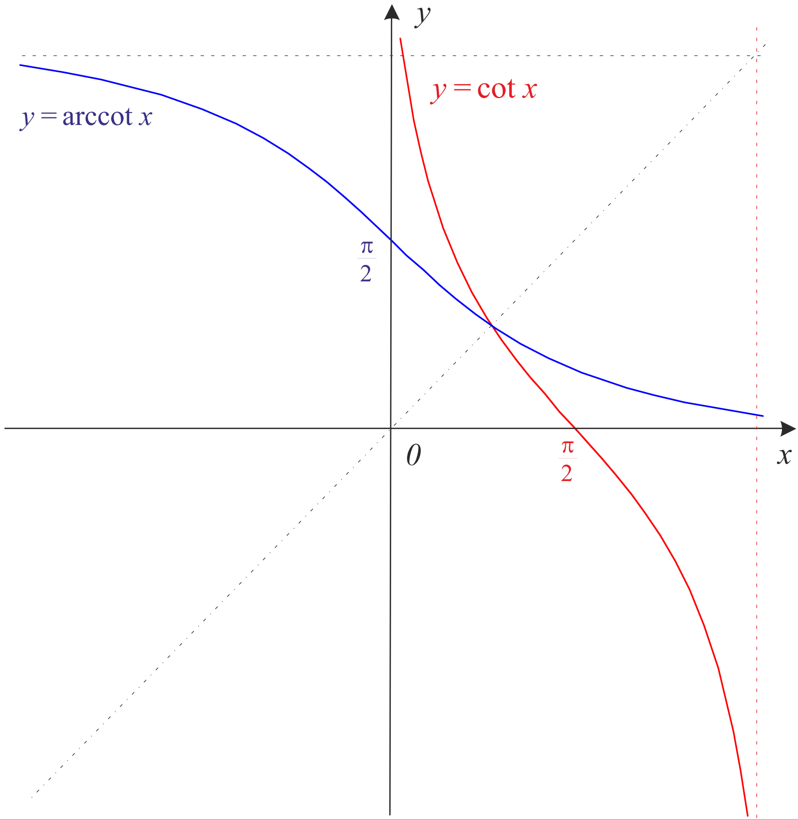
$$f: \; y = \operatorname{arccotg} x, \quad \mathbf{D}(f) = \mathbb{R}\,,$$

is defined as an inverse function to $\cot x$ on the interval $(0, \pi)$, i.e.,

$$y = \operatorname{arccotg} x \quad \Longleftrightarrow \quad x = \cot y, \quad y \in (0, \pi).$$



y
$y = \text{tg}\, x$
$\frac{\pi}{2}$
$y = \text{arctg}\, x$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
x
$-\frac{\pi}{2}$



y
y = cot x
y = arccot x
π/2
0
π/2
x

Hyperbolic functions

Functions **hyperbolic sinus and hyperbolic cosinus**,

$$f:\ y = \sinh x, \quad \mathbf{D}(f) = \mathbb{R},$$

$$f:\ y = \cosh x, \quad \mathbf{D}(f) = \mathbb{R},$$

are defined by the relations

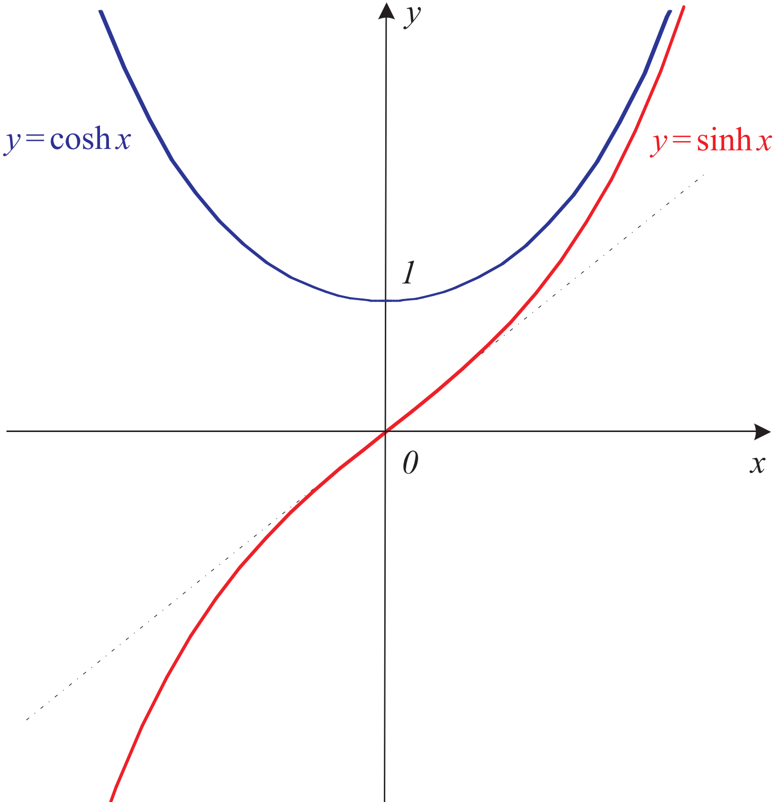
$$\sinh x = \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2}, \qquad \cosh x = \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2}$$

From the definition of $\sinh x$ and $\cosh x$ if follows that:

$$\cosh^2 x - \sinh^2 x = 1,$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y,$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y\,.$$


y = cosh x
y = sinh x
y
x
1
0

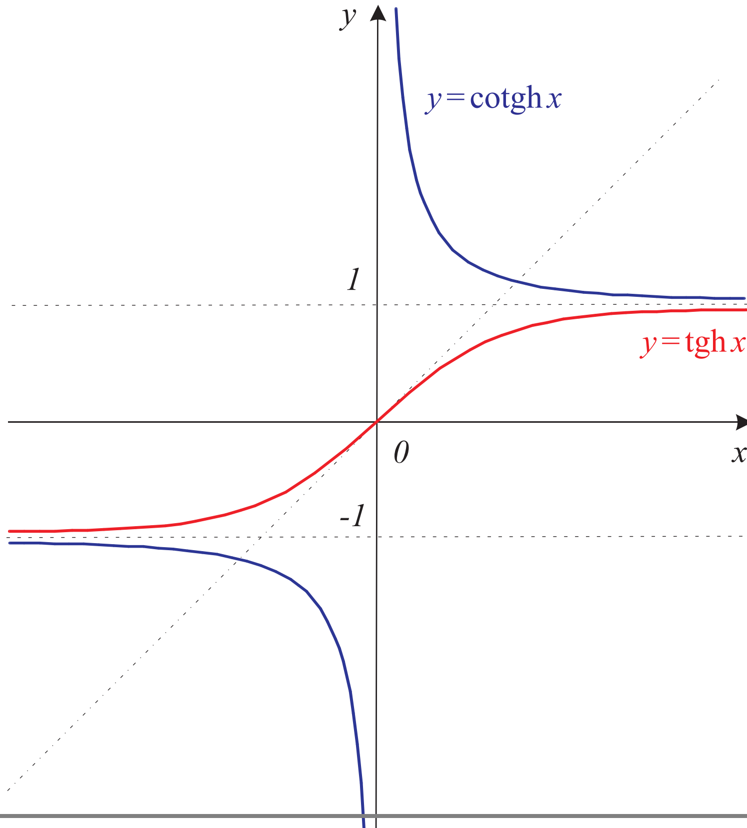
Functions **hyperbolic tangent and hyperbolic cotangent**,

$$f: \; y = \tanh x, \quad \mathbf{D}(f) = \mathbb{R},$$

$$f: \; y = \coth x, \quad \mathbf{D}(f) = \mathbb{R} \setminus \{0\},$$

are defined by

$$\tan x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}}, \qquad \coth x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}}$$


y
y = cotgh x
1
y = tgh x
0
x
-1

Hyperbolometric functions

Function **argument hyperbolic sinus**,

$$f:\ y = \operatorname{argsinh} x, \quad \mathbf{D}(f) = \mathbb{R},$$

is defined as a function inverse to hyperbolic sinus:

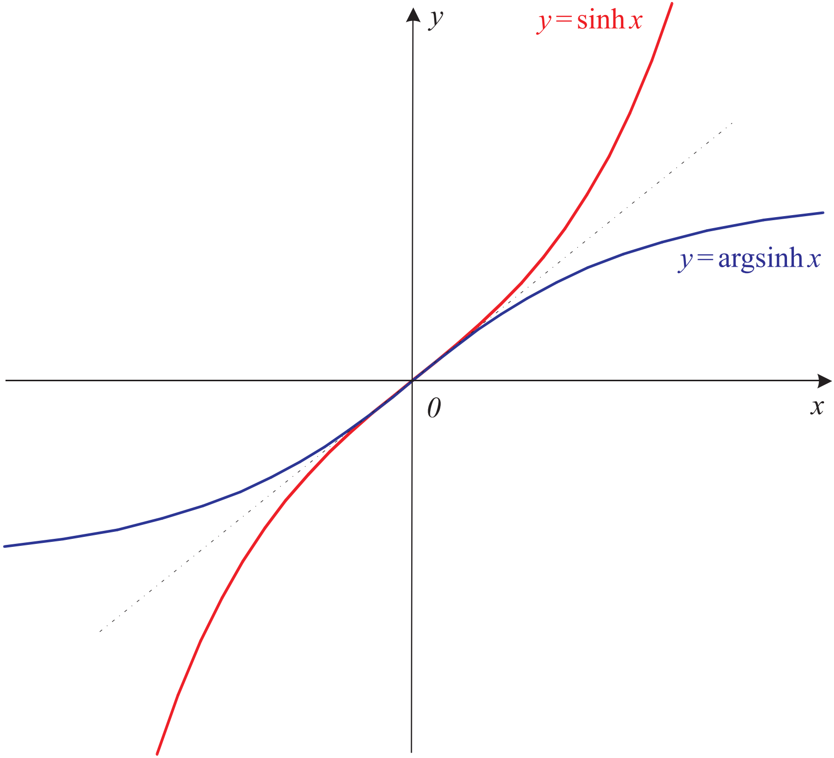
$$y = \operatorname{argsinh} x \iff x = \sinh y, \quad y \in \mathbb{R},$$

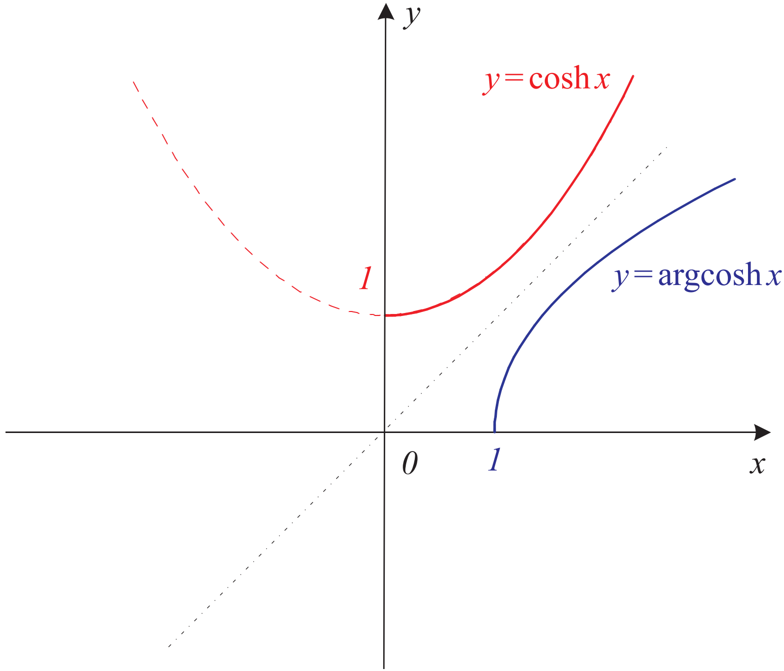
Function **argument hyperbolic cosinus**,

$$f:\ y = \operatorname{argcosh} x, \quad \mathbf{D}(f) = [1, \infty)\,,$$

is defined as a function inverse to hyperbolic cosinus:

$$y = \operatorname{argcosh} x \iff x = \cosh y, \quad y \in [0, \infty)$$


y
y = sinh x
y = argsinh x
0
x


y
y = cosh x
1
y = argcosh x
0
1
x

Function **argument hyperbolic tangent**,

$$f: \; y = \tanh x, \quad \mathbf{D}(f) = (-1, 1),$$

is defined as an inverse function to hyperbolic tangens:

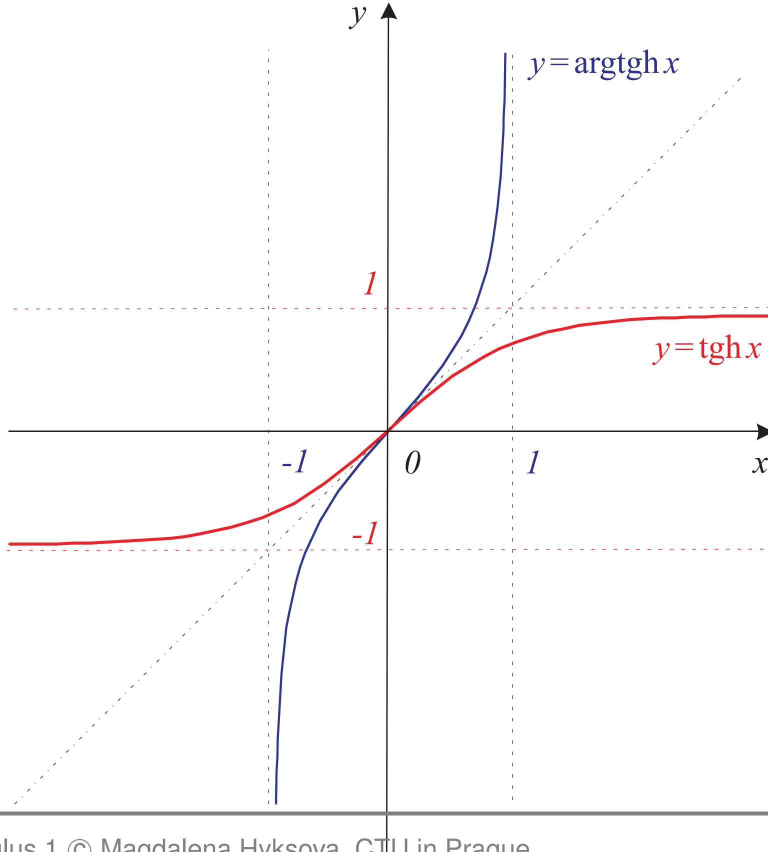
$$y = \operatorname{argtgh} x \iff x = \tanh y, \quad y \in \mathbb{R}$$

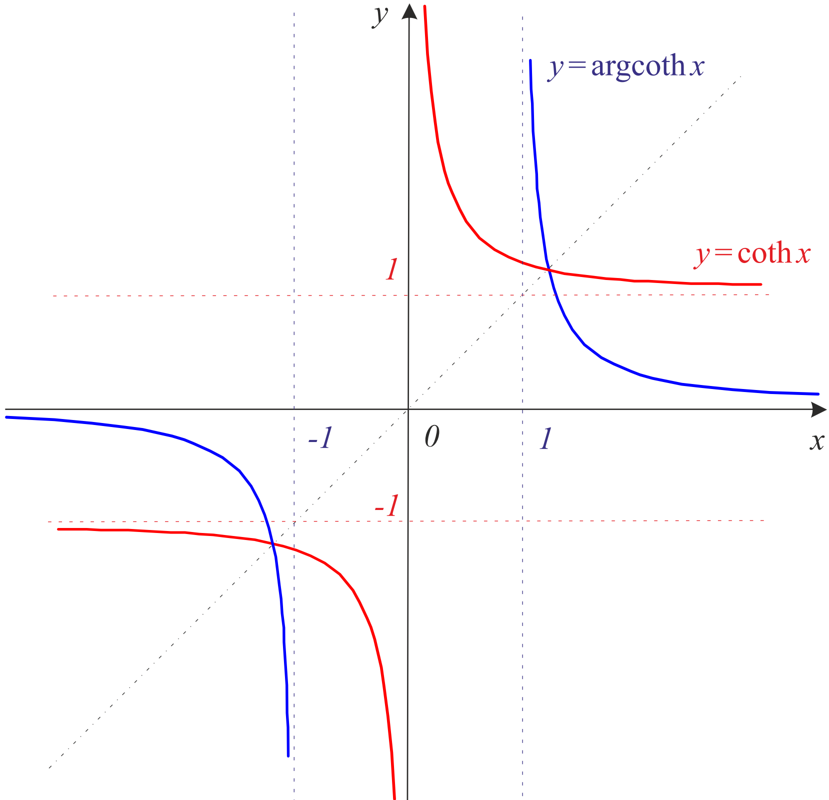
Function **argument hyperbolic cotangent**,

$$f: \; y = \coth x, \quad \mathbf{D}(f) = (-\infty, -1) \cup (1, +\infty),$$

is defined by

$$y = \operatorname{argcotgh} x \iff x = \coth y, \quad y \in \mathbb{R} \setminus \{0\}$$


y
y = argtgh x
1
y = tgh x
-1
0
1
x
-1


y
y = argcoth x
y = coth x
1
-1
0
1
x
-1