BANKRUPTCY GAMES

Contested Garment:

Two hold a garment; one claims it all, the other claims half. Then the one is awarded $\frac{3}{4}$, *the other* $\frac{1}{4}$. (Talmud)

Claims: $d_1 = 1,\ d_2 = 1/2$

Estate: $E < d_1 + d_2 + \cdots + d_n$

Solution: $x = (x_1, x_2);\ x_1 + x_2 = E$

(x_i is the amount assigned to claimant i)

Solution prescribed by the CG (contested garment) principle:

$$x_i = \frac{E - (E - d_1)_+ - (E - d_2)_+}{2} + (E - d_j)_+\,, \qquad \text{where} \ \ (\alpha)_+ := \text{Max}\,(\alpha, 0)$$

Bankruptcy Problem: $(E, d),\ \ d = (d_1, d_2, \ldots d_n)$

Claimants: $1, 2, \ldots, n$

Debts: $d_1 \geq 0,\ d_2 \geq 0,\ \ldots,\ d_n \geq 0,$

Estate: $E < d_1 + d_2 + \cdots + d_n$

Solution: $x = (x_1, x_2, \ldots x_n);\ x_1 + x_2 + \cdots + x_n = E$

(x_i is the amount assigned to claimant i)

Consistent solution: for all $i \neq j$ the division of $x_i + x_j$ prescribed by the contested garment principle for claims d_i, d_j is (x_i, x_j)

Theorem (Aumann, Maschler, 1985):
Each bankruptcy problem has a unique consistent solution.

Bankruptcy game corresponding to the bankruptcy problem (E, d) :

$$v_{E,d}(S) := (E - d(N \setminus S))_+\,, \qquad \text{where} \ \ (\alpha)_+ := \text{Max}\,(\alpha, 0)$$

Theorem (Aumann, Maschler, 1985): The consistent solution of a bankruptcy problem is the nucleolus of the corresponding game.

The standard solution of a 2-person game v :

$$x_i = \frac{v(1,2) - v(1) - v(2)}{2} + v(i)$$

equivalently: $x_1 + x_2 = v(1,2),\ x_1 - x_2 = v(1 - v(2))$
In words, the standard solution gives each player i the amount $v(i)$ that he can assure himself, and divides the remainder equally between the two players. The nucleolus, kernel and the Shapley value of a 2-person game all coincide with its standard solution.

CONTESTED GARMENT GAME

Characteristic function: $v(1) = \frac{1}{2},\ v(2) = 0, v(1,2) = 1$

Core:

$$a_1 \geq \frac{1}{2},\ a_2 \geq 0,\ a_1 + a_2 = 1 \qquad \rightsquigarrow \qquad a_1 \in \langle\frac{1}{2}, 1\rangle,\ a_2 = 1 - a_1$$

Shapley Value:

$$H_1 = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}\,, \qquad H_2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Nucleolus:

$$e(\{1\}, \boldsymbol{a}) = \frac{1}{2} - a_1 \quad e(\{2\}, \boldsymbol{a}) = 0 - a_2 \quad e(\{1,2\}, \boldsymbol{a}) = 1 - a_1 - a_2 = 0$$

$$\begin{aligned} \boldsymbol{e}(\boldsymbol{a}) &= \left(-a_2, \frac{1}{2} - a_1, 0\right) = \left(-a_2, a_2 - \frac{1}{2}\,, 0\right) \\ \boldsymbol{f}(\boldsymbol{a}) &= \left(0, -a_2, a_2 - \frac{1}{2}\right) \qquad \textbf{iff} \quad 0 \leq a_2 \leq \tfrac{1}{4} \\ &= \left(0, a_2 - \frac{1}{2}\,, -a_2\right) \qquad \textbf{iff} \quad \tfrac{1}{4} \leq a_2 \leq 1 \end{aligned}$$

$$\rightsquigarrow \quad \textbf{minimize} \quad \rightsquigarrow \quad a_1 = \frac{3}{4}\,,\ a_2 = \frac{1}{4}$$

Talmud:

$$a_1 = \frac{1}{2} + \frac{1 - \frac{1}{2}}{2} = \frac{3}{4}\,,\ a_2 = 0 + \frac{1}{4} = \frac{1}{4}$$

Nash: **Status Quo:** $(u_0, v_0) = (\frac{1}{2}\,, 0)$

$$g(u, v) = \left(u - \frac{1}{2}\right) v = \left(u - \frac{1}{2}\right)(1 - u) = \frac{3}{2}u - u^2 - \frac{1}{2} = h(u)$$

$$h'(u) = \frac{3}{2} - 2u = 0 \qquad \rightsquigarrow \qquad u = \frac{3}{4}\,,\ v = \frac{1}{4}$$

BANKRUPTCY GAME – 3 CLAIMANTS

$d_1 = 100,\ d_2 = 200,\ d_3 = 300$

E=100 $\quad v(1) = v(2) = v(3) = v(1,2) = v(1,3) = v(2,3) = 0,\ v(1,2,3) = 100$

Imputations: $a_i \geq 0,\ a_1 + a_2 + a_3 = 100$

Shapley: $\quad \boldsymbol{H} = \left(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right)$

Nucleolus:

$$\begin{aligned} \boldsymbol{e}(\boldsymbol{a}) &= (-a_1, -a_2, -a_3, -a_1 - a_2, -a_1 - a_3, -a_2 - a_3, 100 - a_1 - a_2 - a_3) \\ \boldsymbol{f}(\boldsymbol{a}) &= (0, -a_i, -a_j, -a_k, \dots) = (0, -a_i, -a_j, -100 + a_i + a_j, \dots) \end{aligned}$$

$$\rightsquigarrow \quad \textbf{minimize} \quad \rightsquigarrow \quad a_1 = a_2 = a_3 = \frac{100}{3}$$

Proportional division of E : $\quad \left(\frac{100}{6}, \frac{200}{6}, \frac{300}{6}\right)$

$$\left(0, -\frac{100}{6}, -\frac{200}{6}, -\frac{300}{6}, \dots\right) >_{lex} \left(0, -\frac{100}{3}, -\frac{100}{3}, -\frac{100}{3}, \dots\right)$$

E=200 $\quad v(1) = v(2) = v(3) = v(1,2) = v(1,3) = 0,\ v(2,3) = 100, v(1,2,3) = 200$

Imputations: $a_i \geq 0,\ a_1 + a_2 \geq 0,\ a_1 + a_3 \geq 0,\ a_2 + a_3 \geq 100,\ a_1 + a_2 + a_3 = 200$

Shapley: $\quad \boldsymbol{H} = \left(\frac{100}{3}, \frac{250}{3}, \frac{250}{3}\right)$

Nucleolus:

$$\begin{aligned} \boldsymbol{e}(\boldsymbol{a}) &= (-a_1, -a_2, -a_3, -a_1 - a_2, -a_1 - a_3, 100 - a_2 - a_3, 0) \\ \boldsymbol{f}(\boldsymbol{a}) &= (0, -a_1, a_1 - 100, -a_2, -a_3, -a_1 - a_2, -a_1 - a_3) \end{aligned}$$

$$\rightsquigarrow \quad \textbf{minimize} \quad \rightsquigarrow \quad a_1 = 50,\ a_2 = a_3 = 75$$

E=300 $\quad v(1) = v(2) = v(3) = v(1,2) = 0,\ v(1,3) = 100,\ v(2,3) = 200, v(1,2,3) =$ 300

Imputations: $a_i \geq 0,\ a_1 + a_2 \geq 0,\ a_1 + a_3 \geq 0,\ a_2 + a_3 \geq 100,\ a_1 + a_2 + a_3 = 200$

Shapley: $\quad \boldsymbol{H} = \left(\frac{100}{3}, \frac{250}{3}, \frac{250}{3}\right)$

Nucleolus:

$$\begin{aligned} \boldsymbol{e}(\boldsymbol{a}) &= (-a_1, -a_2, -a_3, -a_1 - a_2, 100 - a_1 - a_3, 200 - a_2 - a_3, 0) \\ \boldsymbol{f}(\boldsymbol{a}) &= (0, -a_1, a_1 - 100, -a_2, a_2 - 200, -a_3, a_3 - 300) \end{aligned}$$

$$\rightsquigarrow \quad \textbf{minimize} \quad \rightsquigarrow \quad a_1 = 50,\ a_2 = 100,\ a_3 = 150$$