Contested Garment:

Two hold a garment; one claims it all, the other claims half. Then the one is awarded \( \frac{3}{4} \), the other \( \frac{1}{4} \). (Talmud)

Claims: \( d_1 = 1, \  d_2 = 1/2 \)

Estate: \( E < d_1 + d_2 + \cdots + d_n \)

Solution: \( x = (x_1, x_2); \  x_1 + x_2 = E \)

\( (x_i \text{ is the amount assigned to claimant } i) \)

Solution prescribed by the CG (contested garment) principle:

\[
x_i = \frac{E - (E - d_1)_+ - (E - d_2)_+}{2} + (E - d_j)_+, \quad \text{where} \ (\alpha)_+ := \text{Max} (\alpha, 0)
\]

Bankruptcy Problem: \( (E, d), \ d = (d_1, d_2, \ldots d_n) \)

Claimants: \( 1, 2, \ldots, n \)

Debts: \( d_1 \geq 0, \ d_2 \geq 0, \ldots,\ d_n \geq 0, \)

Estate: \( E < d_1 + d_2 + \cdots + d_n \)

Solution: \( x = (x_1, x_2, \ldots x_n); \ x_1 + x_2 + \cdots + x_n = E \)

\( (x_i \text{ is the amount assigned to claimant } i) \)

Consistent solution: for all \( i \neq j \) the division of \( x_i + x_j \) prescribed by the contested garment principle for claims \( d_i, d_j \) is \( (x_i, x_j) \)

**Theorem** (Aumann, Maschler, 1985):
Each bankruptcy problem has a unique consistent solution.

**Bankruptcy game corresponding to the bankruptcy problem** \( (E, d) \):

\[
v_{E,d}(S):= (E - d(N \setminus S))_+, \quad \text{where} \ (\alpha)_+ := \text{Max} (\alpha, 0)
\]

**Theorem** (Aumann, Maschler, 1985): The consistent solution of a bankruptcy problem is the nucleolus of the corresponding game.

**The standard solution** of a 2-person game \( v \):

\[
x_i = \frac{v(1,2) - v(1) - v(2)}{2} + v(i)
\]

equivalently: \( x_1 + x_2 = v(1,2), \ x_1 - x_2 = v(1) - v(2) \)

In words, the standard solution gives each player \( i \) the amount \( v(i) \) that he can assure himself, and divides the remainder equally between the two players. The nucleolus, kernel and the Shapley value of a 2-person game all coincide with its standard solution.
CONTESTED GARMENT GAME

Characteristic function: \( v(1) = \frac{1}{2}, \ v(2) = 0, \ v(1, 2) = 1 \)

Core:

\[
a_1 \geq \frac{1}{2}, \ a_2 \geq 0, \ a_1 + a_2 = 1 \quad \implies \quad a_1 \in \left( \frac{1}{2}, 1 \right), \ a_2 = 1 - a_1
\]

Shapley Value:

\[
H_1 = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}, \quad H_2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}
\]

Nucleolus:

\[
e(\{1\}, a) = \frac{1}{2} - a_1 \quad e(\{2\}, a) = 0 - a_2 \quad e(\{1, 2\}, a) = 1 - a_1 - a_2 = 0
\]

\[
e(a) = \left(-a_2, \frac{1}{2} - a_1, 0\right) = \left(-a_2, a_2 - \frac{1}{2}, 0\right)
\]

\[
f(a) = \left(0, -a_2, a_2 - \frac{1}{2}\right) \quad \text{iff} \quad 0 \leq a_2 \leq \frac{1}{4}
\]

\[
= \left(0, a_2 - \frac{1}{2}, -a_2\right) \quad \text{iff} \quad \frac{1}{4} \leq a_2 \leq 1
\]

\[
\implies \text{minimize} \quad \implies a_1 = \frac{3}{4}, \ a_2 = \frac{1}{4}
\]

Talmud:

\[
a_1 = \frac{1}{2} + 1 - \frac{1}{2} = \frac{3}{4}, \ a_2 = 0 + \frac{1}{4} = \frac{1}{4}
\]

Nash: Status Quo: \((u_0, v_0) = (\frac{1}{2}, 0)\)

\[
g(u, v) = \left(u - \frac{1}{2}\right) v = \left(u - \frac{1}{2}\right) (1 - u) = \frac{3}{2} u - u^2 - \frac{1}{2} = h(u)
\]

\[
h'(u) = \frac{3}{2} - 2u = 0 \quad \implies \quad u = \frac{3}{4}, \ v = \frac{1}{4}
\]
BANKRUPTCY GAME – 3 CLAIMANTS

d_1 = 100, \; d_2 = 200, \; d_3 = 300

\[ E=100 \]

\[ v(1) = v(2) = v(3) = v(1, 2) = v(1, 3) = v(2, 3) = 0, \; v(1, 2, 3) = 100 \]

Imputations: \( a_i \geq 0, \; a_1 + a_2 + a_3 = 100 \)

Shapley:
\[ H = \left( \frac{100}{3}, \frac{100}{3}, \frac{100}{3} \right) \]

Nucleolus:
\[ e(a) = (-a_1, -a_2, -a_3, -a_1 - a_2, -a_1 - a_3, -a_2 - a_3, 100 - a_1 - a_2 - a_3) \]
\[ f(a) = (0, -a_1, -a_2, -a_3, -a_1 - a_2, -a_1 - a_3) \]
\[ \rightsquigarrow \text{ minimize } \rightsquigarrow a_1 = a_2 = a_3 = \frac{100}{3} \]

Proportional division of \( E \):
\[ \left( 0, -\frac{100}{6}, -\frac{200}{6}, -\frac{300}{6}, \ldots \right) \geq_{\text{lex}} \left( 0, -\frac{100}{3}, -\frac{100}{3}, -\frac{100}{3}, \ldots \right) \]

\[ E=200 \]

\[ v(1) = v(2) = v(3) = v(1, 2) = v(1, 3) = 0, \; v(2, 3) = 100, v(1, 2, 3) = 200 \]

Imputations: \( a_i \geq 0, \; a_1 + a_2 \geq 0, \; a_1 + a_3 \geq 0, \; a_2 + a_3 \geq 100, \; a_1 + a_2 + a_3 = 200 \)

Shapley:
\[ H = \left( \frac{100}{3}, \frac{250}{3}, \frac{250}{3} \right) \]

Nucleolus:
\[ e(a) = (-a_1, -a_2, -a_3, -a_1 - a_2, -a_1 - a_3, 100 - a_2 - a_3, 0) \]
\[ f(a) = (0, -a_1, a_1 - 100, -a_2, -a_3, -a_1 - a_2, -a_1 - a_3) \]
\[ \rightsquigarrow \text{ minimize } \rightsquigarrow a_1 = 50, \; a_2 = a_3 = 75 \]

\[ E=300 \]

\[ v(1) = v(2) = v(3) = v(1, 2) = 0, \; v(1, 3) = 100, \; v(2, 3) = 200, v(1, 2, 3) = 300 \]

Imputations: \( a_i \geq 0, \; a_1 + a_2 \geq 0, \; a_1 + a_3 \geq 0, \; a_2 + a_3 \geq 100, \; a_1 + a_2 + a_3 = 200 \)

Shapley:
\[ H = \left( \frac{100}{3}, \frac{250}{3}, \frac{250}{3} \right) \]

Nucleolus:
\[ e(a) = (-a_1, -a_2, -a_3, -a_1 - a_2, 100 - a_1 - a_3, 200 - a_2 - a_3, 0) \]
\[ f(a) = (0, -a_1, a_1 - 100, -a_2, a_2 - 200, -a_3, a_3 - 300) \]
\[ \rightsquigarrow \text{ minimize } \rightsquigarrow a_1 = 50, \; a_2 = 100, \; a_3 = 150 \]