4 BIMATRIX GAMES

4.1 INTRODUCTION

If the set of players of a normal form game is $Q = \{1, 2\}$ and strategy sets S_1, S_2 are finite, we talk about a **bimatrix game**. Although it is only a special case, we give here fundamental definitions from the previous part once more.

Definition 1. Bimatrix game is a two-player normal form game where

- player 1 has a finite strategy set $S = \{s_1, s_2, \dots, s_m\}$
- player 2 has a finite strategy set $T = \{t_1, t_2, \dots, t_n\}$
- when the pair of strategies (s_i, t_j) is chosen, the payoff to the first player is $a_{ij} = u_1(s_i, t_j)$ and the payoff to the second player is $b_{ij} = u_2(s_i, t_j)$; u_1, u_2 are called **payoff functions**.

The values of payoff functions can be described by a bimatrix:

		Player 2			
	Strategy	t_1	t_2	$\dots$	t_n
	s_1	(a_{11}, b_{11})	(a_{12}, b_{12})	$\dots$	(a_{1n}, b_{1n})
Player 1	s_2	(a_{21}, b_{21})	(a_{22}, b_{22})	$\dots$	(a_{2n}, b_{2n})
	$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$
	s_m	(a_{m1}, b_{m1})	(a_{m2}, b_{m2})	$\dots$	(a_{mn}, b_{mn})

The values of payoff functions can be given separately for particular players:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}.$$

Matrix A is called a **payoff matrix for player 1**, matrix B is called a **payoff matrix for player 2**.

Definition 2. The pair of strategies (s^*, t^*) is called an **equilibrium point**, if and only if

$$\begin{aligned} u_1(s, t^*) &\le u_1(s^*, t^*) \qquad \text{for each } \ s \in S \\ &\text{and} \\ u_2(s^*, t) &\le u_2(s^*, t^*) \qquad \text{for each } \ t \in T. \end{aligned} \tag{4.1}$$

We can easily verify that if $(s^*, t^*) = (s_i, t_j)$ is an equilibrium point, then

- a_{ij} is the **maximum in the column** j of the matrix A : $a_{ij} = \max\limits_{1 \le k \le m} a_{kj}$
- b_{ij} is the **maximum in the row** i of the matrix B : $b_{ij} = \max\limits_{1 \le k \le n} b_{ik}$

☞ ***Example 1.*** Consider a game given by a bimatrix:

Player 2

	Strategy	t_1	t_2
Player 1	s_1	$\boxed{\mathbf{(2,0)}}$	$(2, -1)$
	s_2	$(1, 1)$	$(3, -2)$

The point (s_1, t_1) is apparently an equilibrium: if the second player chose his first strategy t_1 and the first player deviated from his strategy s_1, i.e. chose the strategy s_2, then he would not improve his outcome: he would receive 1 instead of 2. On the other hand, if the first player chose his strategy s_1 a the second player deviated from t_1, then he would not improve his outcome: he would receive -1 instead of 0.

Unfortunately, not in every game an equilibrium point in pure strategies exists:

☞ ***Example 2.*** Consider a bimatrix game:

Player 2

	Strategy	t_1	t_2
Player 1	s_1	$(1, -1)$	$(-1, 1)$
	s_2	$(-1, 1)$	$(1, -1)$

No point is an equilibrium in this game (check particular pairs in the table).

This problem can be removed by introduction of **mixed strategies** that specify **probabilities** with which the players choose their particular pure strategies, i.e. the elements of the sets S, T.

Definition 3. Mixed strategies of players 1 and 2 are the vectors of probabilities $\boldsymbol{p}, \boldsymbol{q}$ for which the following conditions hold:

$$\boldsymbol{p} = (p_1, p_2, \dots p_m); \quad p_i \geq 0, \quad p_1 + p_2 + \cdots + p_m = 1,$$

$$\boldsymbol{q} = (q_1, q_2, \dots q_n); \quad q_i \geq 0, \quad q_1 + q_2 + \cdots + q_n = 1.$$

Here p_j (q_j) expresses the probability of choosing the j-th strategy from the strategy space S (T).

A mixed strategy is therefore again a certain strategy that can be characterized in the following way:

"Use the strategy $s_1 \in S$ with the probability p_1,
. . . ,
use the strategy $s_m \in S$ with the probability p_m."

Similarly for the second player.

Notice that pure strategies correspond to mixed strategies

$$(1, 0, \dots, 0), \ (0, 1, \dots, 0), \ \dots \ (0, 0, \dots, 1).$$

Definition 4. Expected payoffs are defined by the relations:

$$\begin{aligned} &\text{Player 1:} \qquad \pi_1(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij} \\ &\text{Player 2:} \qquad \pi_2(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j b_{ij} \end{aligned} \tag{4.2}$$

Theorem 1. *In mixed strategies, every bimatrix game has at least one equilibrium point.*

4.2 SEARCHING EQUILIBRIUM STRATEGIES

4.2.1 Iterated Elimination of Dominated Strategies

In some cases it is possible to eliminate obviously bad, so-called dominated strategies:

Definition 5. The strategy $s_k \in S$ of the player 1 is called **dominating** another strategy $s_i \in S$ if, for each strategy $t \in T$ of the player 2 we have:

$$u_1(s_k, t) \geq u_1(s_i, t);$$

dominating strategy of the second player is defined in the same way.

If there remains the only element in the bimatrix after an iterated elimination of dominated strategies, it is the desired equilibrium point. If there remain more elements, we have at least a simpler bimatrix.

We illustrate the process by the following example.

☛ ***Example 3.*** Consider the bimatrix game:

Player 2

	Strategy	t_1	t_2	t_3
	s_1	$(1, 0)$	$(1, 3)$	$(3, 0)$
Player 1	s_2	$(0, 2)$	$(0, 1)$	$(3, 0)$
	s_3	$(0, 2)$	$(2, 4)$	$(5, 3)$

The strategy s_2 of the first player is dominated by the strategy s_3, because he receives more when he chooses s_3 than when he chooses s_2, whatever strategy is chosen by the second player. Similarly the strategy t_3 of the second player is dominated by the strategy t_2. Since the rational player 1 will not choose the dominated strategy s_2 and the rational player 2 will not choose the dominated strategy t_3, the decision is reduced in this way:

Player 2

	Strategy	t_1	t_2
Player 1	s_1	$(1, 0)$	$(1, 3)$
	s_3	$(0, 2)$	$\boxed{(2, 4)}$

Strategy t_1 is dominated by the strategy t_2, the second player therefore chooses t_2. The first player now decides between the values in the second column of the bimatrix, and since $1 < 2$, he chooses s_3. Hence an equilibrium point of the game is (s_3, t_2) – think over the fact that in the original matrix, one-sided deviation from the equilibrium strategy do not bring an improvement to the "deviant".

☛ ***Example 4.*** An investor wants to build two hotels. One of them will be called Big (abbreviated as B); gaining the order for it will bring the profit of 15 million to the building firm. The second hotel will be called Small (abbreviated as S); gaining the order for it will bring the profit of 9 million to the building firm.
There are two firms competing for the orders, we will denote them 1 and 2. None of them has a potential to build both hotels in full. Each firm can offer the building of one hotel or the cooperation on both of them. The investor has to realize the building by force of these firms and he decides on the base of a sealed-bid method. The rules for splitting the orders according to offers are the following:

1. If only one firm bids for a contract on a hotel, it receives it all.
2. If two firms bid for a contract on the same hotel and none of them bids for the second one, the investor offers the cooperation to both firms on both hotels, the profits are split fifty-fifty.
3. If one firm bids for a contract on the whole building of a hotel and the second firm offers a collaboration, than the firm that bids the whole building receives 60% and the second 40% in the case of B, and 80% versus 20% in the case of S. On the building of the other hotel the firms will collaborate fifty-fifty (including splitting the profit).

In every case the total profit of $15 + 9 = 24$ milion is split between the firms. Find optimal strategies for both firms.

The payoffs corresponding to particular strategies can be represented by a bimatrix:

		Firm 2		
	Strategy	Big	Small	Cooperation
	Big	$\boxed{(12, 12)}$	$(15, 9)$	$(13,5; 10,5)$
Firm 1	Small	$(9, 15)$	$(12, 12)$	$(14,7; 9,3)$
	Cooperation	$(10,5; 13,5)$	$(9,3; 14,7)$	$(12, 12)$

Strategy "cooperation" is dominated by the strategy "big" for both firms, we can therefore elliminate the third row and the third column. Now we have a bimatrix with only two rows and two columns (strategies "big" and "small"). Here the strategy "small" is dominated by the strategy "big" and can therefore be elliminated, too. There remains the only point: the strategy pair ("big","big") and it is an equilibrium point.

4.2.2 Best Reply Correspondence

According to the definition, equilibrium strategies s^*, t^* forming the equilibrium point (s^*, t^*) are mutually best replies in the sense that if the first player chooses his equilibrium strategy s^*, then the second player can not improve his outcome by deviating from t^*, similarly the first player can not improve his outcome by deviating from s^*, provided the second player chooses t^*.

More exactly:

Definition 6. Best reply of player 1 to the strategy t of player 2 is defined as the set

$$R_1(t) = \{s^* \in S;\ u_1(s^*, t) \geq u_1(s, t) \text{ for each } s \in S\}\,. \tag{4.3}$$

Similarly **best reply of player 2 to the strategy s of player 1** is defined as the set

$$R_2(s) = \{t^* \in T;\ u_2(s, t^*) \geq u_2(s, t) \text{ for each } t \in T\}\,. \tag{4.4}$$

If the strategy sets consist of two elements for both players, the sets R_1 and R_2 are curves in the plain – so-called **reaction curves**.

Theorem 2. *(s^*, t^*) is an equilibrium point if and only if it is*

$$s^* = R_1(t^*) \qquad \textit{and} \qquad t^* = R_2(s^*).$$

Proof. According to the definition, $s^* = R_1(t^*)$ if and only if for each $s \in S$ it is

$$u_1(s^*, t^*) \geq u_1(s, t^*), \tag{4.5}$$

similarly $t^* = R_2(s^*)$ if and only if for each $t \in T$ it is:

$$u_2(s^*, t^*) \geq u_1(s^*, t). \tag{4.6}$$

Inequalities (4.5), (4.6) correspond to the conditions for an equilibrium point given in Definition 2.

Searching an equilibrium point, we can construct **reaction curves** and find their **intersection**.

☛ ***Example 5.*** For the game from the example 1, the best reply of player 1 to the strategy t_1 of player 2 is the strategy s_1, i.e.

$$R_1(t_1) = s_1,$$

similarly the best reply of player 1 to the strategy t_2 is the strategy s_2, i.e.

$$R_1(t_2) = s_2.$$

Similarly for the best replies of player 2:

$$R_2(s_1) = t_1, \qquad R_2(s_2) = t_1.$$

In this case, it is easy to find the pair of strategies that are mutually best replies: it is the pair (s_1, t_1) which is, according to the above discussion, an equilibrium point of the game.

☛ ***Example 6.*** For the game from the example 2 we have

$$R_1(t_1) = s_1, \quad R_1(t_2) = s_2.$$

$$R_2(s_1) = t_2, \quad R_2(s_2) = t_1.$$

In this case, no pair of strategies consists from mutually best replies. As it was already mentioned, it is necessary to consider mixed strategies.

Player 1 will choose his first strategy s_1 with the probability p and the second strategy s_2 with the probability $1-p$. Player 2 will choose his first strategy t_1 with the probability q and the second strategy t_2 with the probability $1-q$:

Player 2

	Strategy	t_1	t_2	
Player 1	s_1	$(1,-1)$	$(-1,1)$	$\ldots\ldots$ $\boldsymbol{p}$
	s_2	$(-1,1)$	$(1,-1)$	$\ldots\ldots$ $\boldsymbol{1-p}$
		$\boldsymbol{q}$	$\boldsymbol{1-q}$	

Expected payoffs to particular players are the following:

$$\begin{aligned}
\pi_1(p,q) &= 1\cdot p\cdot q - 1\cdot p\cdot(1-q) - 1\cdot(1-p)\cdot q + 1\cdot(1-p)\cdot(1-q)\\
&= pq - p + pq - q + pq + 1 - p - q + pq = 4pq - 2p - 2q + 1\\
&= p(4q-2) - 2q + 1
\end{aligned}$$

$$\begin{aligned}
\pi_2(p,q) &= -1\cdot p\cdot q + 1\cdot p\cdot(1-q) + 1\cdot(1-p)\cdot q - 1\cdot(1-p)\cdot(1-q)\\
&= -pq + p - pq + q - pq - 1 + p + q - pq = -4pq + 2q + 2p - 1\\
&= q(-4p+2) + 2p - 1
\end{aligned}$$

Now we will search best replies of the player 1 to various choices of probability q :

If $\boxed{0 \leq q < \frac{1}{2},}$ then for a fixed value of q, $\pi_1(p, q)$ is a linear function with the negative slope, which is therefore **decreasing**. Maximum of this function occurs for the least possible value of p, i.e. for $p = 0$; in this case it is: $R_1(q) = 0$.

If $\boxed{q = \frac{1}{2},}$ then $\pi_1(p, \frac{1}{2}) = 0$ is a **constant function** for which each value is maximal and minimal – player 1 is therefore indifferent between both strategies, $R_1(\frac{1}{2}) = \langle 0, 1\rangle$.

If $\boxed{\frac{1}{2} < q \leq 1,}$ then for a fixed value of q, $\pi_1(p, q)$ is a linear function with the positive slope, which is therefore **increasing**. Maximum occurs for the greatest possible value of p, i.e. for $p = 1$; in this case it is $R_1(q) = 1$.

On the whole:

$$R_1(q) = \begin{cases} 0 & \text{for} \quad 0 \leq q < \frac{1}{2} \\ \langle 0, 1\rangle & \text{for} \quad q = \frac{1}{2} \\ 1 & \text{for} \quad \frac{1}{2} < q \leq 1 \end{cases}$$

Similarly for player 2:

$$R_2(p) = \begin{cases} 1 & \text{for} \quad 0 \leq p \leq \frac{1}{2} \\ \langle 0, 1\rangle & \text{for} \quad p = \frac{1}{2} \\ 0 & \text{for} \quad \frac{1}{2} \leq p \leq 1 \end{cases}$$

The curves can be represented in the plain:

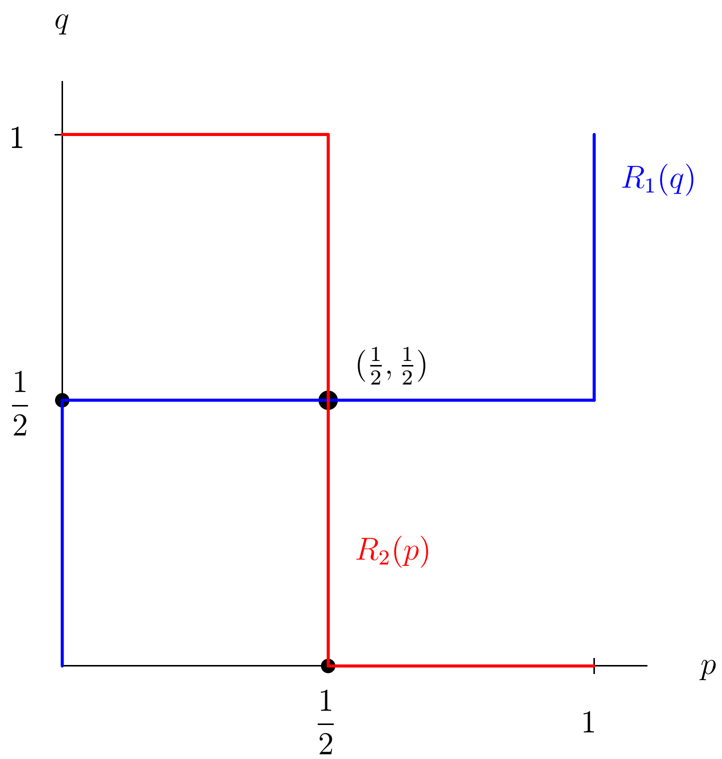

FIG. 4.1: REACTION CURVES FOR THE GAME FROM EXAMPLE 2

Equilibrium point is therefore

$$\left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right).$$

Provided the players follow these strategies, the expected payoff is 0 for each of them.

A Usefull Principle for Equilibrium Strategies Search:

A mixed strategy $s^* = (p_1, \ldots, p_m)$ is the best reply to t^* if and only if each of pure strategies that occur in s^* with positive probability is the best reply to t^*.

The player who optimalizes using a mixed strategy is therefore indifferent among all pure strategies that occur in a given mixed strategy with positive probabilities.

(Notice that if for example a pure strategy s_1 would be more advantageous in a given situation than s_2, than whenever the player would be about to use s_2, it would be better to use s_1 – we would not have an equilibrium point.)

☛ ***Example 7.*** Consider a bimatrix game

Player 2

	Strategy	t_1	t_2
Player 1	s_1	$(4,-4)$	$(-1,-1)$
	s_2	$(0,1)$	$(1,0)$

Expected values for particular players are the following:

$$\begin{aligned}\pi_1(p,q) &= 4pq - p(1-q) + 0 + (1-p)(1-q)\\ &= p(6q-2) - q + 1\end{aligned}$$

$$\begin{aligned}\pi_2(p,q) &= -4pq - p(1-q) + (1-p)q + 0\\ &= q(-4p+1) - p\end{aligned}$$

Now we will search best replies of the player 1 to various choices of probability q :

If $\boxed{0 \le q < \frac{1}{3},}$ then for a fixed value of q, $\pi_1(p,q)$ is a linear function with the negative slope, which is therefore **decreasing**. Maximum of this function occurs for the least possible value of p, i.e. for $p = 0$; in this case it is: $R_1(q) = 0$.

If $\boxed{q = \frac{1}{3},}$ then $\pi_1(p, \frac{1}{3}) = \frac{2}{3}$ is a **constant function** for which each value is maximal and minimal – player 1 is therefore indifferent between both strategies, $R_1(\frac{1}{3}) = \langle 0, 1\rangle$.

If $\boxed{\frac{1}{3} < q \le 1,}$ then for a fixed value of q, $\pi_1(p,q)$ is a linear function with the positive slope, which is therefore **increasing**. Maximum occurs for the greatest possible value of p, i.e. for $p = 1$; in this case it is $R_1(q) = 1$.

On the whole:

$$R_1(q) = \begin{cases} 0 & \text{for} \quad 0 \le q < \frac{1}{3} \\ \langle 0, 1 \rangle & \text{for} \quad q = \frac{1}{3} \\ 1 & \text{for} \quad \frac{1}{3} < q \le 1 \end{cases}$$

Similarly for player 2:

$$R_2(p) = \begin{cases} 1 & \text{for} \quad 0 \le p \le \frac{1}{4} \\ \langle 0, 1 \rangle & \text{for} \quad p = \frac{1}{4} \\ 0 & \text{for} \quad \frac{1}{4} \le p \le 1 \end{cases}$$

The curves can be represented in the plain:

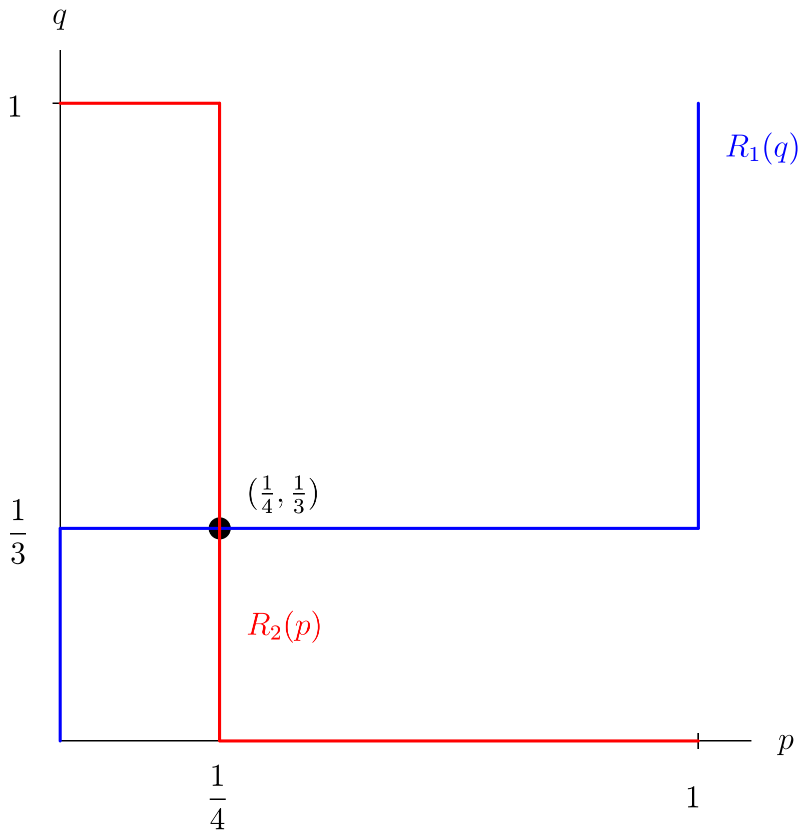

FIG. 4.2: REACTION CURVES FOR THE GAME FROM EXAMPLE 7

Equilibrium point is therefore

$$\left(\left(\frac{1}{4}, \frac{3}{4} \right), \left(\frac{1}{3}, \frac{2}{3} \right) \right).$$

Provided the players follow these strategies, the expected payoff to the first player will be $\frac{2}{3}$ and to the second one $-\frac{1}{4}$.

On the base of the above principle, searching an equilibrium point can be significantly simplified:

If q is an equilibrium strategy of player 2, player 1 has to be indifferent between his pure strategies s_1, s_2 (compare Fig. 4.3). Hence the expected payoffs have to be the same for both pure strategies of player 1 for the mixed strategy $(q, 1-q)$ of player 2:

$$\pi_1(1, q) = 4q - (1-q) = 0 + (1-q) = \pi_1(0, q)$$

$$6q = 2 \quad \Rightarrow \quad q = \frac{1}{3}$$

Similarly, in order p is an equilibrium strategy of player 1, player 2 has to be indifferent between his pure strategies t_1, t_2 (compare Fig. 4.3). Hence the expected payoffs have to be the same for both pure strategies of player 2 for the mixed strategy $(p, 1-p)$ of player 1:

$$\pi_2(p, 1) = -4p + (1-p) = -p + 0 = \pi_2(p, 0)$$

$$1 = 4p \quad \Rightarrow \quad p = \frac{1}{4}$$

In this way we have come to the same equilibrium point: $\left(\left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$.

Guidelines for Computing Mixed Strategies Equilibria:

- *Consider a bimatrix game G with matrices A, B.*
- *Expected payoffs given by (4.2) can be expressed as functions of variables $p_1, p_2, \ldots p_{m-1}$; $q_1, q_2, \ldots q_{n-1}$, namely due to relations*

$$p_m = 1 - (p_1 + p_2 + \cdots + p_{m-1}), \qquad q_m = 1 - (q_1 + q_2 + \cdots + q_{n-1}).$$

- *Consider a system of equations:*

$$\begin{aligned} \frac{\partial \pi_1}{\partial p_i} &= 0 \quad \textit{for all} \quad i = 1, 2, \ldots, m-1 \\ \frac{\partial \pi_2}{\partial q_j} &= 0 \quad \textit{for all} \quad j = 1, 2, \ldots, n-1 \end{aligned} \tag{4.7}$$

Any solution of the system (4.7),

$$\boldsymbol{p} = (p_1, p_2, \ldots, p_m); \qquad \boldsymbol{q} = (q_1, q_2, \ldots, q_n).$$

where

$$p_i \geq 0, \qquad q_j \geq 0 \qquad \textit{for all} \quad i, j$$

$$p_1 + p_2 + \cdots + p_{m-1} \leq 1, \qquad q_1 + q_2 + \cdots + q_{n-1} \leq 1,$$

is a **mixed strategies equilibrium**.

☛ ***Example 8.*** Find equilibrium strategies in the game **Scissors-paper-stone**:

		Player 2			
		Scissors	Paper	Stone	
Player 1	Scissors	(0,0)	(1,-1)	(-1,1)	p_1
	Paper	(-1,1)	(0,0)	(1,-1)	p_2
	Stone	(1,-1)	(-1,1)	(0,0)	$1-p_1-p_2$
		q_1	q_2	$1-q_1-q_2$	

Expected payoffs:

$$\pi_1(\boldsymbol{p},\boldsymbol{q}) = p_1q_2 - p_1(1-q_1-q_2) - p_2q_1 + p_2(1-q_1-q_2) + (1-p_1-p_2)q_1 - (1-p_1-p_2)q_2$$

$$\begin{aligned}\pi_1(\boldsymbol{p},\boldsymbol{q}) &= 3p_1q_2 - 3p_2q_1 - p_1 + p_2 + q_1 - q_2 \\ \pi_2(\boldsymbol{p},\boldsymbol{q}) &= -3p_1q_2 + 3p_2q_1 + p_1 + p_2 - q_1 + q_2\end{aligned}$$

$$\frac{\partial\pi_1}{\partial p_1} = 3q_2 - 1 = 0 \qquad\qquad \frac{\partial\pi_2}{\partial p_1} = 3p_2 - 1 = 0$$

$$\frac{\partial\pi_1}{\partial p_2} = -3q_2 + 1 = 0 \qquad\qquad \frac{\partial\pi_2}{\partial p_2} = -3p_1 + 1$$

Solution: $p_1 = p_2 = q_1 = q_2 = \frac{1}{3}$, therefore

$$(\boldsymbol{p},\boldsymbol{q}) = \left((\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}),\ (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3})\right)$$

4.2.3 Games with More Equilibrium Points

So far we have met examples where only one equilibrium point existed, whatever in pure or mixed strategies. But often more equilibrium points exist and a question arises, which of them shall be considered as optimal.

Let us start with several definitions.

Definition 7. Let $(\boldsymbol{p},\boldsymbol{q})$ be an equilibrium point of the bimatrix game G such that

$$\pi_1(\boldsymbol{p},\boldsymbol{q}) \geq \pi_1(\boldsymbol{r},\boldsymbol{s}) \qquad \text{and} \qquad \pi_2(\boldsymbol{p},\boldsymbol{q}) \geq \pi_2(\boldsymbol{r},\boldsymbol{s})$$

for any equilibrium point $(\boldsymbol{r},\boldsymbol{s})$ of the game G. Then $(\boldsymbol{p},\boldsymbol{q})$ is called **dominating equilibrium point of the game** G.

If there exists the only equilibrium point, it is obviously dominating.

Example 9. Consider a bimatrix game

$$\begin{pmatrix} \underline{(3,2)} & (-1,1) \\ (-2,0) & \boxed{(6,5)} \end{pmatrix}$$

There exist two equilibrium points in pure strategies, namely (s_1, t_1) and (s_2, t_2). The second of them dominates the first one, since for the payoffs we have: $6 > 3$ a $5 > 2$. Hence for both players it is the most advantageous to choose the second strategy.

Example 10. Consider a bimatrix game

Player 2

Player 1		t_1	t_2	t_3
	s_1	(-2,-2)	(-1,0)	$\boxed{(8,6)}$
	s_2	(0,-1)	$\underline{(5,5)}$	(0,0)
	s_3	$\boxed{(8,6)}$	(0,-1)	(-2,-2)

In this game there exist three pure equilibrium points: (s_1, t_3), (s_2, t_2), (s_3, t_1). The first and last ones are dominating. Nevertheless, since the players have no chance to make a deal, it can happen that they choose strategy pairs (s_1, t_1) or (s_3, t_3) and reach the worst possible outcomes.

Definition 8. Let $(\boldsymbol{p}^{(j)}, \boldsymbol{q}^{(j)})$, $j \in J$, are equilibrium points of a bimatrix game G. These points are called **interchangeable**, if the value of payoff functions $\pi_1(\boldsymbol{p}, \boldsymbol{q})$ and $\pi_2(\boldsymbol{p}, \boldsymbol{q})$ do not change when we put any $\boldsymbol{p}^{(j)}$, $j \in J$ instead of $\boldsymbol{p}$ and any $\boldsymbol{q}^{(j)}$, $j \in J$ instead of $\boldsymbol{q}$.

Example 11. Let us change the bimatrix from the previous example:

Player 2

Player 1		t_1	t_2	t_3
	s_1	$\boxed{(8,6)}$	(-1,0)	$\boxed{(8,6)}$
	s_2	(0,-1)	$\underline{(5,5)}$	(0,0)
	s_3	$\boxed{(8,6)}$	(0,-1)	$\boxed{(8,6)}$

All dominating equilibrium points (s_1, t_1), (s_1, t_3), (s_3, t_1) and (s_3, t_3) are now interchangeable and the trouble from the previous example can not occur. This fact motivates the following definition.

Definition 9. All interchangeable dominating equilibrium points of a given game G are called **optimal points of the game** G. If there exist such points in a game, the game is called **solvable**.

☛ ***Example 12*** – **Battle of the Buddies.**
Consider a married couple in which the partners have a bit different opinions of how to spend a free evening: the wife prefers the visit of a box match, the husband prefers the visit of a football match. Payoffs are represented by the bimatrix

		Husband	
		Box	Football
Wife	Box	$(2,1)$	$(0,0)$
	Football	$(0,0)$	$(1,2)$

The game has two equilibrium points in pure strategies, namely

$$(box,\ box),\ (football,\ football),$$

and another equilibrium point in mixed strategies, $\left((\frac{1}{3},\frac{2}{3}),\ (\frac{2}{3},\frac{1}{3})\right)$, with the corresponding values of expected payoffs $\left(\frac{2}{3},\frac{2}{3}\right)$.

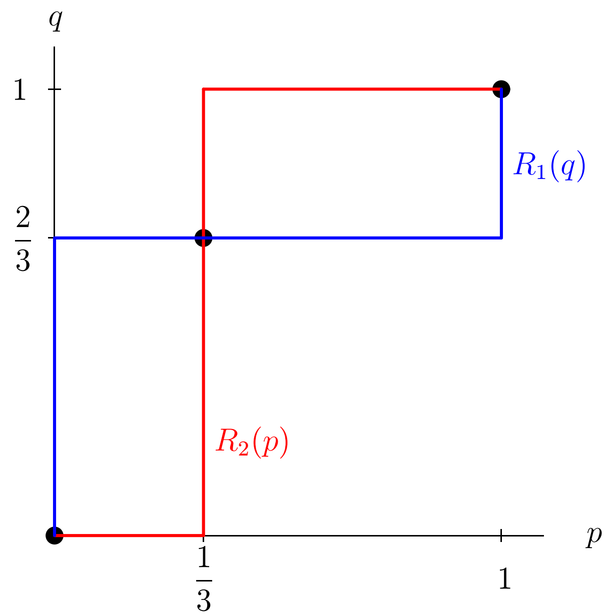

Fig. 4.3: Reaction Curves for the Battle of the Buddies

These values can easily be found by solving the equation:

$$\pi_1(\boldsymbol{p},\boldsymbol{q}) = p_1(3q_1-1)+1-q_1 = q_1(3p_1-2)-2p_1+2 = \pi_2(\boldsymbol{p},\boldsymbol{q}).$$

Since no equilibrium point is dominated, this game is not solvable in the sense of the definition 9.

4.3 EQUILIBRIUM STRATEGIES IN BIOLOGY

4.3.1 Introduction

In biology, game theory is the main tool for the investigation of conflict and cooperation of animals and plants. As for zoological applications, the theory of games is used for the analysis, modeling and understanding the fight, cooperation and communication of animals, coexistence of alternative traits, mating systems, conflict between the sexes, offspring sex ratio, distribution of individuals in their habitats, etc. Among botanical applications we can find the questions of seed dispersal, seed germination, root competition, nectar production, flower size, sex allocation, etc.

Apparently the first work where a game-theoretical approach was used (although the author was not aware of it), was Fisher's book [6] from 1930; it came about in the connection with the evolution of the sex ratio and with the sexual selection. The first explicit and conscious application of game theory in biology is contained in Lewontin's paper on evolution [9] from 1961. Nevertheless, Lewontin considered species to play a game against the nature, while the modern theory pictures members of a population as playing games against each other and studies the population dynamics and equilibria which can arise; this second approach was foreshadowed in the mentioned Fisher's book [6] and developed – including explicit terminology of game theory – by W. D. Hamilton [7], [8], R. L. Trivers [13] and some others. However, these works remained isolated and they did not stir up any wider interest.

A historical milestone is represented by the short but "epoch-making" paper *The Logic of Animal Conflict* by J. Maynard Smith and G. R. Price. This treatise, published in 1973, stimulated a great deal of successful works and applications of game theory in evolutionary biology; the development of the following decade was summarized in Maynard Smith's book *Evolution and the Theory of Games* [11]. Not only proved game theory to provide the most satisfying explanation of the theory of evolution and the principles of behavior of animals and plants in mutual interactions, it was just *this field* which turned out to provide the most promising applications of the theory of games at all. Is this a paradox? How is it possible that the behavior of animals or plants prescribed on the base of game-theoretical models agree with the action observed in the nature? Can a fly or a fig tree, for example, be a rational decision-maker who evaluates all possible outcomes and by the tools of game theory selects his optimal strategy? How is it possible that even the less developed the thinking ability of an organism is, the better game theory tends to work?

As the birth of game theory von Neumann's paper [**?**] from 1928 is usually considered; nevertheless, the origin of the theory as a "real" mathematical discipline is connected with the book [12] of John von Neumann and Oscar Morgenstern, first published in 1944; only then game theory became widely known.

4.3.2 The Game of Genes

The explanation is simple: the *players* of the game are not taken to be the organisms under study, but the *genes* in which the instinctive behavior of these organisms is coded. The *strategy* is then the *behavioral phenotype*, i.e. the behavior preprogrammed by the genes – the specification of what an individual will do in any situation in which it may find itself; the *payoff function* is a *reproductive fitness*, i.e. the measure of the ability of a gene to survive and spread in the genotype of the population in question. The main *solution concept* of this model the *evolutionary stable strategy* which is defined as *a strategy such that, if all the members of a population adopt it, no mutant strategy can invade.* In certain specific situations, this somewhat vague concept is expressed more precisely. For example, the pairwise contests in an infinite asexual population can be modelled by a two-player normal form game and the evolutionary stable strategy I is a strategy satisfying the following conditions:

$$\begin{array}{lll} \text{for all } \ J \neq I & \text{either} & u_1(I,I) > u_1(J,I) \\ & \text{or} & u_1(I,I) = u_1(J,I) \quad \text{and} \quad u_1(I,J) > u_1(J,J); \end{array} \tag{4.8}$$

in biological terms, the value of the payoff function $u_1(s_1, s_2)$ is the expected change of fitness of an individual adopting a strategy s_1 against an opponent adopting s_2 (fitness can be measured for example by the number of offspring).

In short, to understand the basic principles of so-called "genocentric" conception of the evolution, it suffices to imagine that about four thousand million years ago a remarkable molecule was formed by accident that was able to create copies of itself – we will call it the *replicator.* It started to spread its copies; during the replication sometimes a mistake or *mutation* occurred, some of these mutations led to replicators that were more successful in mutual contests and in reproduction. Some of them could discover how to break up molecules of rival varieties chemically, others started to construct for themselves containers, vehicles for their continued existence. Such replicators survived, that had better and more effective *"survival machines"*. Generation after generation, the *survival machines* of genes, i.e. the organisms controlled by the genes, compete in mutual contests; the genes that choose the best strategy for their *machine* spread themselves and step by step their *learning* proceeds. The result is that these *machines* act in the same way as game theory would calculate – instead of the calculation they have come to the equilibrium strategies by gradual adaptation and natural selection.

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[11], p. 204.

Due to the symmetry, the pair of strategies (I, I) form an equilibrium point.

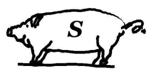
It shall be noted that there exist good field measurements of costs and benefits in nature, which have been plugged into particular models; see e.g. [3]. Hence the payoff can really be measured quantitatively.

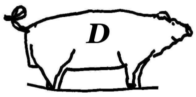
Equilibrium Strategies and Learning

An illustrative analogy of the slow evolution process is the learning of an individual who finds himself repeatedly in the same conflict situation during his relatively short lifetime. Apparently the most interesting experiment was made in 1979 by B. A. Baldwin and G. B. Meese with the Skinner sty: there is the snout-lever at one end of the sty, the food dispenser at the other. Pressing the lever causes pureing the food down a chute. Baldwin and Meese placed two domestic pigs into this sty. Such couple always settles down into a stable *dominant/subordinate* hierarchy. Which pig will press the lever and run across the sty and which one will be sitting by the food trough? The situation is schematically illustrated in Fig. 1.

The strategy *"If dominant, sit by the food trough; if subordinate, work the lever"* sounds sensible, but would not be stable. The subordinate pig would soon give up pressing the lever, for the habit would never be rewarded. The reverse strategy *"If dominant, work the lever; if subordinate, sit by the food trough"* would be stable – even though if it has the paradoxical result that the subordinate pig gets most of the dominant pig when he charges up from the other end of the sty. As soon as he arrives, he has no difficulty in tossing the subordinate pig out of the trough. As long as there is a crumb left to reward him, his habit of working the lever will persist.

Using the theory of games, we would model the situation by a *bimatrix game*:





	Press the lever		*Sit by the trough*
Press the lever	$(8,-2)$	$\rightarrow$	$\boxed{(\mathbf{6},\mathbf{4})}$
			$\uparrow$
Sit by the trough	$(10,-2)$	$\rightarrow$	$(0,0)$

Rational players would come to the equilibrium strategies in the following way. For the second player – the subordinate pig – the first strategy is dominated by the second one and can be therefore eliminated. The first player – the dominant pig – presumes the rationality of his opponent and hence decides between the profit of 0 or 6 units in the second column, which leads him to the choice of the first strategy. Indeed, the couple of strategies (*press the lever*, *sit by the trough*) is an equilibrium point.

For more details see [4], pp. 286–287, and the original paper [2] by Baldwin and Meese.

We consider the profit from the whole ration to the extent of 10 utility units, the loss caused by the labor connected with pressing the lever and running -2 units and the amount of food which the subordinate pig manages to eat before he is tossed out by the dominant one, 4 units (these units were chosen at random – from the strategic point of view nothings changes when the labor is evaluated with an arbitrary negative number, the waiting subordinate pig receives nonnegative number of units and nonnegative number of units remains for the dominant one.

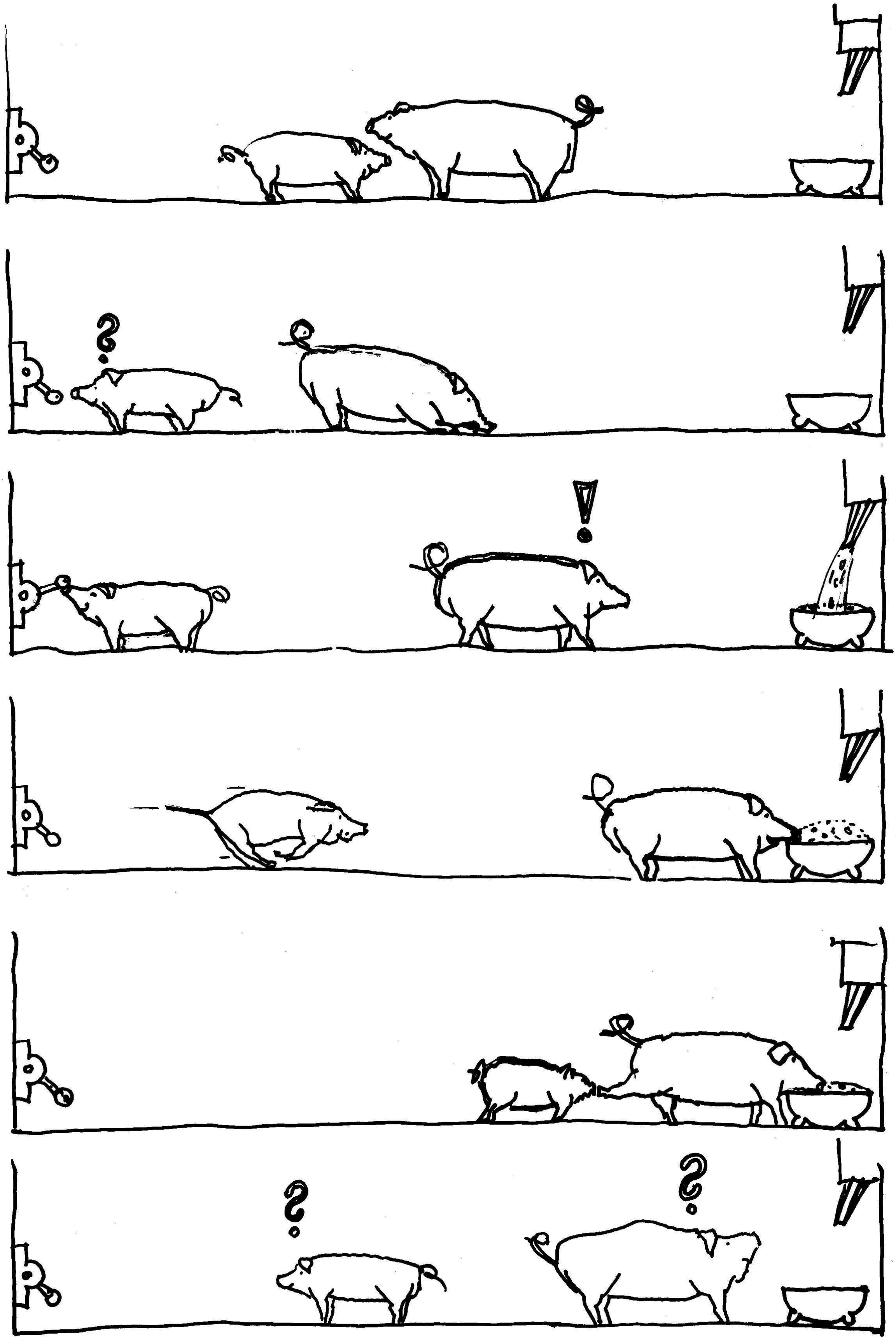
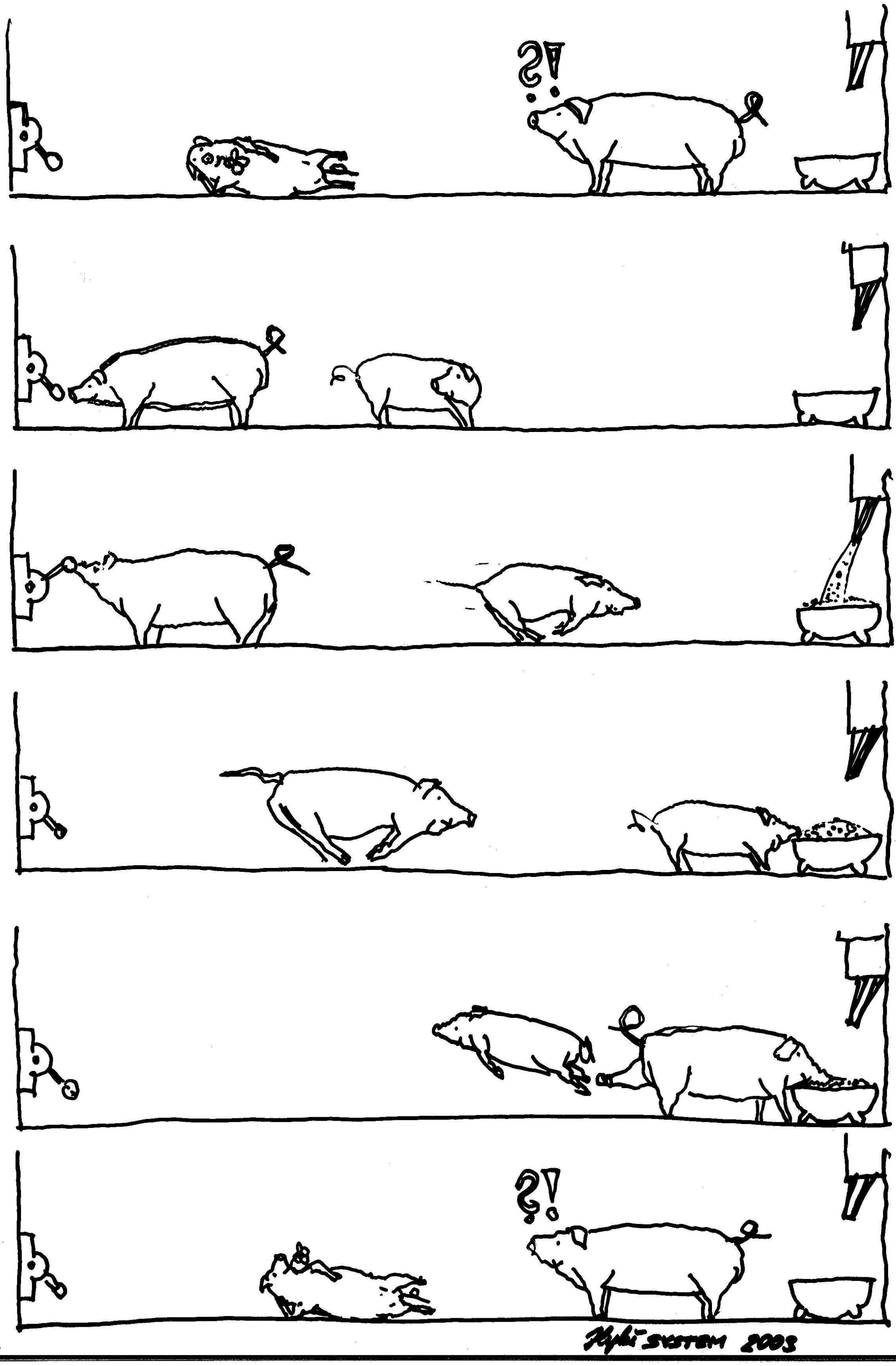


Fig. 4.4: Skinner Box Experiment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4.3.3 Conclusion

There are many interesting examples in the domain of biology which can be used not only to motivate the students or liven up the lessons, but also to show how evolution really works – and getting to the heart of the evolution principles and hence to the heart of our lives is an exceptionally exciting experience. We ask the kind reader to take this contribution as the invitation to the reading of the books [4] and [5] by R. Dawkins, the book [11] by Maynard Smith – the cited literature can be a good beginning of a more in-depth study.

Several remarks should be added in conclusion. First, the modern theory of evolution based on "games of selfish genes" is much more satisfactory than the former theory which considered the organisms acting "for the good of the species". In addition to all items solved by the "group selection theory" (e.g. the existence of "limited war" strategies and altruism among non-relatives), the genocentric one can explain most of contradictions and questions that the former theory can not answer.

What is mainly surprising but auspicious, is the fact that the gene's-eye view of evolution provides the solid explanation of the existence of a real altruism in the nature: not only among relatives, but also among non-relative individuals who interact in a long run – many instinctively acting species shall be taken as shining examples by us humans, often governed by negative emotions. For example, a functioning reciprocal altruism underlies the regular alternation of the sex roles in the hermafrodite sea bass, the reciprocal help between mails of pavian anubi to fight off an attacker during the time one of them is mating, or the blood-sharing by the great mythmakers vampires (the bats eating the cattle blood). In the words of R. Dawkins, ... *vampires could form the vanguard of a comfortable new myth, a myth of sharing, mutualistic cooperation. They could herald the benignant idea that, even with selfish genes at the helm, nice guys can finish first.*

[4], p. 233.

4.4 REPEATED GAMES – PRISONER'S DILEMMA

4.4.1 Examples

☛ ***Example 13*** – **Prisoner's Dilemma 1**

One of the interpretations of the conflict that is called *Prisoner's dilemma* is the following one:

It is 1930's. In the Soviet Union at that time a conductor travels by train to Moscow, to the symphony orchestra concert. He studies the score and concentrates on the demanding performance. Two KGB agents are watching him, who – in their ignorance – think that the score is a secret code. All conductors efforts to explain that it is yet Tchajkovskij are absolutely hopeless. He is arrested and imprisoned. The second day our couple of agents visit him with the words: "You have better speak. We have found your comrade Tchajkovskij and he is already speaking ..."

Two innocent people, one because he studied a score and the second because his name was coincidentally Tchajkovskij, find themselves in prison, faced the following problem: if both of them bravely keep denying, despite physical and psychical torture, they will be sent to Gulag for three years, then they will be released. If one of them confesses the fictive espionage crime of them both, and the second one keeps denying, then the first one will get only one year in Gulag, while the second one 25. If both of them confess, they will be sent to Gulag for 10 years.

The situation can be described by the bimatrix:

		Tchajkovskij	
		Deny	Confess
Conductor	Deny	$(-3,-3)$	$(-25,-1)$
	Confess	$(-1,-25)$	$\boxed{(-10,-10)}$

This situation is called *dilemma* because commonly it would be the most convenient for both to keep denying and go to Gulag for three years. The problem is that they have no chance to make a deal – and even if they had a chance to make a deal, there is a danger of comrade's confessing – whatever under a press or a temptation to take advantage of a shorter sentence. And even if both of them were solidary, each of them can think about the other that he falls prey to the temptation or a torture and confesses – hence he is in danger of 25 year sentence which is even much worse than 10 years. Both therefore choose the second strategy and confess.

Indeed, the strategy *confess* dominates the strategy *deny* and the pair

$$(confess,\ confess)$$

is the only equilibrium point in the game.

☛ ***Example 14*** – **Prisoner's Dilemma 2**

More generally, prisoner's dilemma is a name for every situation of the type (compare the example 13):

Player 2

Player 1		Cooperate	Defect
	Cooperate	(*reward, reward*)	(*sucker, temptation*)
	Defect	(*temptation, sucker*)	(*punishment, punishment*)

where

$$sucker < punishment < reward < temptation.$$

Cooperation can express whatever – the strategy pair (*cooperate, cooperate*) corresponds to mutually solidary action.

Where Prisoner's Dilemma Occurs – Some More Examples

- **Building the Sewage Water Treatment Plant** (two big hotels by one mountain lake):
  - *Cooperate* = build the purify facility
  - *Defect* = do not build it
  - *Reward* = pure water attracts tourists – customers, profits increase, nevertheless, we had to invest a certain sum of money
  - *Temptation* = take advantage of the purify facility of the second hotel and save on the investment
  - *Punishment* = polluted water discourages tourists, the profit decreases to zero
- **Duopolists:**
  - *Cooperate* = collude on the optimal total production (corresponding to monopoly)
  - *Defect* = break the deal
  - *Reward* = the highest total profit
  - *Temptation* = produce somewhat more at the expense of the second duopolist
  - *Punishment* = less profit for both

- **Removing the Parasites:**
  - *Cooperate* = mutual removing of parasites
  - *Defect* = have removing done by the comrade but do not return the favor
  - *Reward* = I will be free of parasites, nevertheless I will pay it by removing yours
  - *Temptation* = I will be free of parasites without paying it back
  - *Punishment* = all are full of parasites which is much worse than a slight effort to remove the other's parasites
- **Public Transportation:**
  - *Cooperate* = pay the fare
  - *Defect* = do not pay
  - *Reward* = public transportation runs, I can use it, nevertheless I have to pay a certain sum every month.
  - *Temptation* = use the public transportation but do not pay
  - *Punishment* = (almost) nobody pays, the public transportation is dissolved, I have to pay a taxi which is much more expensive than the original fare payment
- **Television Licence Fee:**
  - *Cooperate* = pay
  - *Defect* = do not pay
  - *Reward* = public service broadcast works, I can watch it, but I have to pay some small sum of money
  - *Temptation* = do not pay and watch
  - *Punishment* = (almost) nobody pays, the broadcast is dissolved
- **Battle:**
  - *Cooperate* = fight
  - *Defect* = hide
  - *Reward* = victory but also a risk of injury
  - *Temptation* = victory without a risk of injury
  - *Punishment* = the enemy wins without any fighting
- **Nuclear Armament:**
  - *Cooperate* = disarm
  - *Defect* = arm
  - *Reward* = the world without nuclear threat
  - *Temptation* = to be the only one armed
  - *Punishment* = all arm, pay much money for it, moreover a danger threats

4.4.2 Repeated Prisoner's Dilemma

In the case of infinite or indeterminate time horizon, *Cooperate* is not necessarily irrational:

☛ ***Example 15*** – **Prisoner's Dilemma 3**

Consider the following variant of Prisoner's dilemma:

Player 2

Player 1		Cooperate	Defect
	Cooperate	$(3,3)$	$(0,5)$
	Defect	$(5,0)$	$(1,1)$

Imagine that the game will be repeated with the probability of $2/3$ in each round that the next round occurs, too.

When both players cooperate, the expected payoff for each of them is:

$$\pi_C = 3 + 3 \cdot \tfrac{2}{3} + 3 \cdot (\tfrac{2}{3})^2 + 3 \cdot (\tfrac{2}{3})^3 + \cdots + 3 \cdot (\tfrac{2}{3})^n + \cdots$$

Strategy in repeated game is a complete plan how the player will act in the whole course of the game in all possible situations in which he can find himself.

For example, consider a strategy **Grudger**:

Cooperates until the second has defected, after that move defects forever.

When two Grudgers meet in a game, they will cooperate all the time and each of them receives the value $\pi_G = \pi_C$.

It can easily be proven that the pair of strategies

$$(\textit{Grudger, Grudger})$$

is an **equilibrium point** of the game in question.

Consider a **Deviant** who deviates from the **Grudger** strategy played with Grudger. In some round this Deviant defects, although the Grudger has cooperated (this can also happen in the first round). Let this deviation occurs first in the round $n+1$. Since the Deviant plays with the Grudger, in the next round the opponent chooses his strategy **defect** and holds on it forever. The Deviant can not therefore obtain more than

$$\pi_D = 3 + 3 \cdot \tfrac{2}{3} + 3 \cdot (\tfrac{2}{3})^2 + 3 \cdot (\tfrac{2}{3})^3 + \cdots + 3 \cdot (\tfrac{2}{3})^{n-1} + 5 \cdot (\tfrac{2}{3})^n + 1 \cdot (\tfrac{2}{3})^{n+1} + \cdots$$

Since

$$\begin{aligned}
\pi_G - \pi_D &= (3-5) \cdot (\tfrac{2}{3})^n + (3-1) \cdot (\tfrac{2}{3})^{n+1} + \cdots + (3-1) \cdot (\tfrac{2}{3})^{n+k} + \cdots \\
&= -2 \cdot (\tfrac{2}{3})^n + 2 \cdot (\tfrac{2}{3})^{n+1} + \cdots + 2 \cdot (\tfrac{2}{3})^{n+k} + \cdots \\
&= (\tfrac{2}{3})^n \left(-2 + 2 \cdot \tfrac{2}{3} \cdot \frac{1}{1 - \frac{2}{3}} \right) = (\tfrac{2}{3})^n \cdot 2 > 0,
\end{aligned}$$

it does not pay to deviate.

Similarly, we can consider the strategy **Tit for Tat**, which begins with cooperation and then plays what its opponent played in the last move. The pair

$$(\textit{Tit for Tat, Tit for Tat})$$

is an equilibrium point, too.

Examples of Strategies in Repeated Prisoner's Dilemma

Always Cooperates

Always Defects

Grudger, Spiteful: Cooperates until the second has defected, after that move defects forever (he does not forgive).

Tit for Tat: begins with cooperation and then plays what its opponent played in the last move (if the opponent defects in some round, Tit for Tat will defect in the following one; to cooperation it responds with cooperation).

Mistrust Tit for Tat: In the first round it defects, than it plays opponent's move.

Naive Prober: Like Tit for Tat, but sometimes, after the opponent has cooperated, it defects (e.g. at random, in one of ten rounds in average).

Remorseful Prober: Like Naive Prober, but he makes an effort to end cycles C–D caused by his own double-cross: after opponent's defection that was a reaction to his unfair defection, he cooperates for one time.

Hard Tit for Tat: Cooperates unless the opponent has defected at least once in the last two rounds.

Gradual Tit for Tat: Cooperates until the opponent has defected. Then, after the first opponent's defection it defects once and twice it cooperates, after the second defection it defects in two subsequent rounds and twice it cooperates, ..., after the n-th opponent's defection it defects in n subsequent rounds and twice it cooperates, etc.

Gradual Killer: In the first five rounds it defects, than it cooperates in two rounds. If the opponent has defected in rounds 6 and 7, than the Gradual Killer keeps defecting forever, otherwise he keeps cooperation forever.

Hard Tit for 2 Tats: Cooperates except the case when the opponent has defected at least in two subsequent rounds in the last three rounds.

Soft Tit for 2 Tats: Cooperates except the case when the opponent has defected in the last two subsequent rounds.

Slow Tit for Tat: Plays C–C, then if opponent plays two consecutive times the same move, plays its move.

Periodically DDC: Plays periodically: Defect–Defect–Cooperate

Periodically SSZ: Plays periodically: Cooperate–Cooperate–Defect

Soft Majority: Cooperates, than plays opponent's majority move, if equal then cooperates.

Hard Majority: Cooperates, than plays opponent's majority move, if equal then defects.

Pavlov: Cooperates if and only if both players opted for the same choice in the previous move, otherwise it defects.

Pavlov P_n: Adjusts the probability of cooperation in units of $1/n$ according to the previous round: when it cooperated with the probability p in the last round, the probability of cooperation in the next round is

$p \oplus \frac{1}{n} = \min(p + \frac{1}{n}, 1)$ if it obtained $R = \textit{reward}$;
$p \ominus \frac{1}{n} = \max(0, p - \frac{1}{n})$ if it obtained $P = \textit{punishment}$;
$p \oplus \frac{2}{n}$ if it obtained $T = \textit{temptation}$;
$\ominus \frac{2}{n}$ if it obtained $S = \textit{sucker}$.

Random: Cooperates with the probability 1/2.

Hard Joss: Plays like Tit for Tat, but it cooperates only with the probability 0.9.

Soft Joss: Plays like Tit for Tat, but it defects only with the probability 0.9.

Generous Tit for Tat: Plays like Tit for Tat, but it after the defection it cooperates with the probability

$$g(R, P, T, S) = \min\left(1 - \frac{T - R}{R - S}, \frac{R - P}{T - P}\right).$$

Better and Better In n-th round it defects with the probability $(1000 - n)/1000$, i.e. the probability of defection is lesser and lesser.

Worse and Worse: In n-th round it defects with the probability $n/1000$, i.e. the probability of defection is greater and greater.

Axelrod's Tournament

In 1981 Robert Axelrod organized a computer tournament in which 15 different strategies for the repeated prisoner's dilemma sent by prominent game theorists fought in pairwise matches, each math consisting of 200 rounds (totally 15×15 matches). The points obtained on the base of the matrix from example 15 were summed.

To a great surprise of all involved, the highest score was won by the simple **Tit for Tat** that was sent by Anatol Rapoport, psychologist and game theory specialist.

In his discussion of the tournament, Axelrod distinguished the following categories of strategies:

- **Nice Strategy** – never defects first (only in retaliation),

  Nasty Strategy – at least sometimes it defects first.

  There were eight nice strategies in the tournament and they took the first eight places (the most successful one obtained 504,5 points which corresponds to 84% of the standard of 600 points, other nice strategies obtained 83,4%–78,6%; the most successful of nasty strategies gained only 66,3%).

- **Forgiving Strategy** – can retaliate but it has a short memory, forgets the old unfairness,

  Non-Forgiving Strategy – never forgets old unfairness, never extricates from a cycle of mutual retaliations – not in the game with a remorseful opponent.

- **Clear Strategy** – it is only interested in its own profit, not in the defeat of the opponent,

  Envious Strategy

- **Retaliatory Strategy** – does not let nasty strategies to exploit it,

  Non-Retaliatory Strategy

The Second Tournament

In the second Axelrod tournament the number of rounds was not strictly given but the tournament went analogically with the evolution based on the natural selection: all strategies gained a payoff corresponding to the number of offspring, the total number of individuals was keeping constant. More successful strategies reproduced at the expense of less successful; after about 1000 generations the stability was reached. The winner was again **Tit for Tat**.

Occurrences of Repeated Prisoner's Dilemma (further examples)

- **Front Linie – Live and Let Live:**
  - *Cooperate* = live and let live
  - *Defect* = kill every man from the opposite side when the opportunity knocks
  - *Reward* = survival of long war years
  - *Temptation* = take advantage of the situation that the opponent is an easy chased and earn for example a medal – it is afterall better to remove the enemy
  - *Punishment* = all are upon the guard all the time...
- **Mutual Help of Males of Baboon Anubi:**
  - *Cooperate* = help the other male drive an enemy away during his mating
  - *Defect* = do not pay the help back
  - *Reward* = successful mating, offspring
  - *Temptation* = take advantage of help but do not pay it back and save the time and effort
  - *Punishment* = less offspring

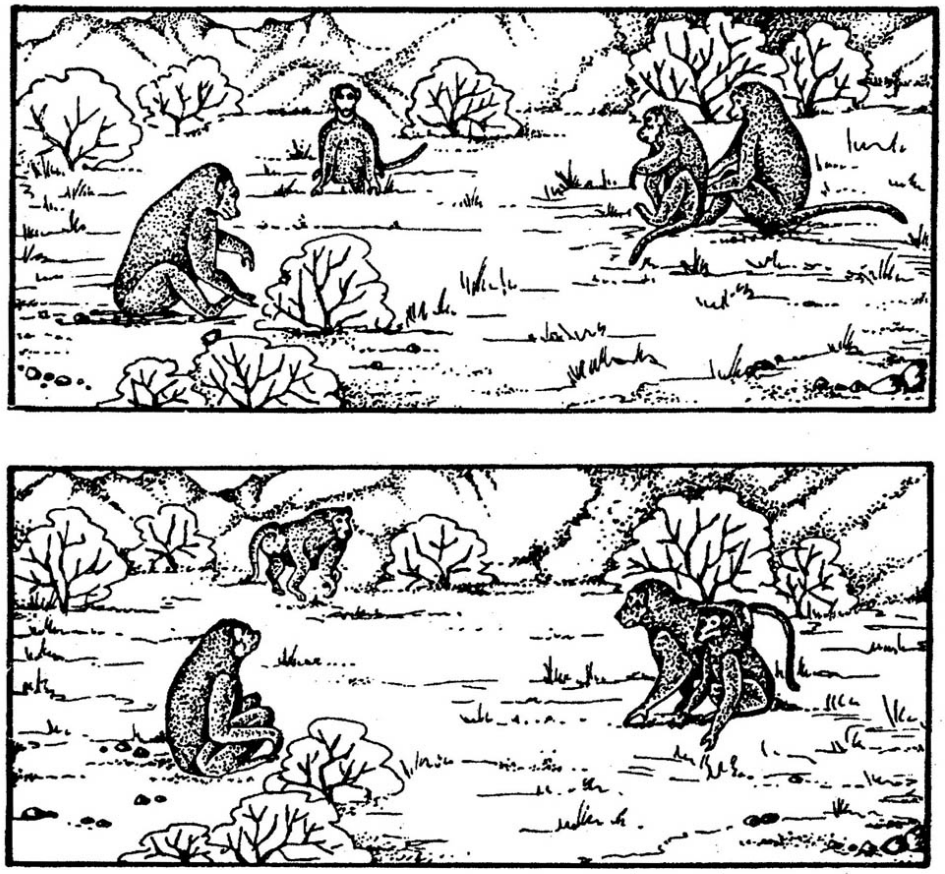

Fig. 4.5: Baboon Anubi

In the nature: the more often a male A supports a male B, the more the male B supports A.

- **Fig Tree and Chalcidflies:**
  - *Cooperate* = balanced ratio of pollinated flowers and flowers with laied eggs inside the fig
  - *Defect* = lay eggs to a greater number of flowers
  - *Reward* = genes spread
  - *Temptation* = lay eggs to a greater number of flowers and hence to encrease the number of offspring
  - *Punishment* = the fig hosting the treacherous Calcidfly family is thrown down and the whole family dies out
- **Sexual Roles Alternating by Hermaphrodite Grouper:**
  - *Cooperate* = if I am a male now, I will became a female the next time
  - *Defect* = became a male again after acting a male
  - *Reward* = living together in harmony, many offspring
  - *Temptation* = repeat an easy male role
  - *Punishment* = the relation breaks down
- **Desmodus Rotundus Vampire (a bat sucking mammal blood) – feeding hungry individuals:**
  - *Cooperate* = after a successful hunt, feed unsuccessful individuals "colleagues"
  - *Defect* = keep all blood
  - *Reward* = long-run successful survival
  - *Temptation* = in the case of need, let the colleagues to feed me, do not share the catch with the others
  - *Punishment* = in the case of unsuccessful hunt, starving out

  In the nature: the individuals that have returned from a unsuccessful hunt are feeded by successful ones, even non-relatives; they recognize each other.

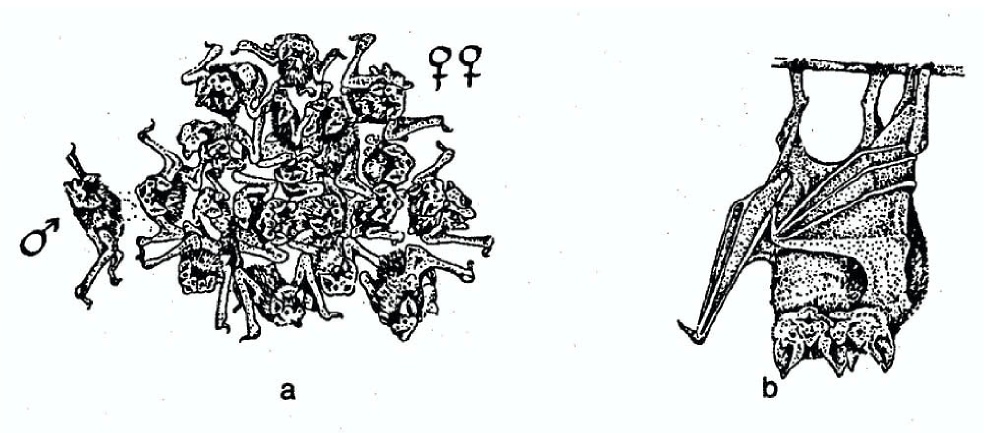

Fig. 4.6: Desmodus Rotundus Vampires

References

[1] Axelrod, R.: *The Evolution of Cooperation.* Basic Books, New York, 1984.

[2] Baldwin, B. A.; Meese, G. B.: *Social Behaviour in Pigs Studied by Means of Operant Conditioning.* Animal Behaivour, **27**(1979), pp. 947–957.

[3] Brockmann, H. J.; Dawkins, R.; Grafen, A.: *Evolutionarily Stable Nesting Strategy in a Digger Wasp.* Journal of Theoretical Biology, **77**(1979), pp. 473–496.

[4] Dawkins, R.: *The Selfish Gene.* Oxford University Press, Oxford, 1976 (Czech translation by V. Kopsý: *Sobecký gen*, Mladá Fronta, Praha, 2003).

[5] Dawkins, R.: *The Blind Watchmaker.* Harlow, Longman, 1986 (Czech translation by T. Grim: *Slepý hodin ař*, Paseka, Praha, 2002).

[6] Fisher, R. A.: *The Genetical Theory of Natural Selection.* Clar. Press, Oxford, 1930.

[7] Hamilton, W. D.: *The Genetical Evolution of Social Behaviour I, II.* Journal of Theoretical Biology **7**(1964), pp. 1–16; 17–52.

[8] Hamilton, W. D.: *Extraordinary Sex Ratios.* Science **156**(1967), pp. 477–488.

[9] Lewontin, R. C.: *Evolution and the Theory of Games.* Journal of Theoretical Biology **1**(1961), pp. 382–403.

[10] Maynard Smith, J. – Price, G. R.: *The Logic of Animal Conflict.* Nature **246**(1973), pp. 15–18.

[11] Maynard Smith, J.: *Evolution and the Theory of Games.* Cambridge, Cambridge University Press, 1982.

[12] von Neumann, J.; Morgenstern, O.: *Theory of Games and Economic Behavior.* Princeton University Press, Princeton, 1944.

[13] Trivers, R. L.: *The Evolution of Reciprocal Altruism.* Quaterly Review of Biology **46**(1971), pp. 35–57.

[14] Williams, G. C.: *Adaptation and Natural Selection.* Princeton University Press, Princeton, 1966.

[15] Williams, G. C.: *Sex and Evolution.* Princeton University Press, Princeton, 1975.

4.5 ANTAGONISTIC GAMES

4.5.1 Two-Player Antagonistic Game

Definition 10. Two-player antagonistic game is a **normal form game with constant sum of payoffs**:

$$\begin{gathered}(Q = \{1, 2\}; S, T; u_1(s,t), u_2(s,t)) \\ u_1(s,t) + u_2(s,t) = \text{const.} \quad \text{for each } (s,t) \in S \times T.\end{gathered} \tag{4.9}$$

When the sum of payoffs in the game (4.9) is equal to **zero**, we simply write

$$u_1(s,t) = u_2(s,t) = u(s,t);$$

the normal form is:

$$(Q = \{1, 2\}; S, T; u(s,t)) \tag{4.10}$$

For **equilibrium strategies** s^*, t^* in a zero-sum game it is:

$$u(s,t^*) \leq u(s^*,t^*) \leq u(s^*,t) \qquad \text{for all } \ s \in S, \ t \in T. \tag{4.11}$$

The value $u(s^*,t^*)$ is called the **value of the game**.

It can be proven that to each two-player constant sum normal form game (4.9) a **zero sum** normal form game can be assigned which is **strategically equivalent** to the original game, i.e. every pair of strategies s, t, that is an equilibrium in the original game is an equilibrium in the corresponding zero sum game, too, and vice versa. More exactly:

Theorem 3. *Let (4.9) be a two-player constant sum game where the sum of payoffs equals K. Then s^*, t^* are equilibrium strategies in the game (4.9) if and only if s^*, t^* are equilibrium strategies in the zero sum game (4.10) where*

$$u(s,t) = u_1(s,t) - u_2(s,t).$$

4.5.2 Matrix Games

Two-player zero-sum games with finite strategy sets

$$S = \{s_1, s_2, \dots s_m\}, \quad T = \{t_1, t_2, \dots t_n\} \tag{4.12}$$

can be described by the **matrix** A,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \tag{4.13}$$

whose elements express payoffs to the first player (payoffs to the second player have always the opposite value).

Equilibrium Strategies in Matrix Games

When a so-called **saddle point**, i.e. an element that represents the **minimum in the row and maximum in the column** exists, then this element gives equilibrium strategies and hence a value of the game, e.g.:

$$A = \begin{pmatrix} 5 & 4 & \boxed{4} & 5 \\ -4 & 5 & 3 & 9 \\ 7 & 8 & -1 & 8 \end{pmatrix} \tag{4.14}$$

To find the saddle point, we can proceed in the following way: for each row we find the minimum (such element represents the least guaranteed profit to the row player) and then we find the maximal element of these minimas (for the corresponding row the least guaranteed profit is the highest), so-called **lower value of the game**,

$$\underline{v} = \max_{i=1,2,\ldots,m} \min_{j=1,2,\ldots,n} a_{ij} \ . \tag{4.15}$$

Similarly, for each column we find the maximum (such element represents the greatest guaranteed loss of the column player) and then we find the minimal element of these maximas (for the corresponding column the greatest guaranteed profit is the least), so-called **upper value of the game**,

$$\overline{v} = \min_{j=1,2,\ldots,n} \max_{i=1,2,\ldots,m} a_{ij} \ . \tag{4.16}$$

In general, it is

$$\underline{v} \leq \overline{v}. \tag{4.17}$$

Theorem 4. *If, for a given matrix A it is $\underline{v} = \overline{v}$, than A has a saddle point.*

The common value $v = \underline{v} = \overline{v}$ is called the **value of the game** and the corresponding strategies of the first and second player are called **maximin** and **minimax strategies**, respectively.

For the above matrix (4.14) we have

$$\begin{array}{cc} A = \begin{pmatrix} 5 & 4 & \boxed{\mathbf{4}} & 5 \\ -4 & 5 & 3 & 9 \\ 7 & 8 & -1 & 8 \end{pmatrix} & \begin{array}{c} \mathbf{4} \\ -4 \\ -1 \end{array} \\ \begin{array}{cccc} 7 & 8 & \mathbf{4} & 9 \end{array} & \end{array}$$

Unfortunately, the saddle point does not always exist, for example:

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \underline{v} = -1, \ \overline{v} = 1; \qquad B = \begin{pmatrix} 0 & -5/2 & -2 \\ -1 & 3 & 1 \end{pmatrix}, \quad \underline{v} = -1, \ \overline{v} = 0.$$

Hence, as before, it is necessary to introduce mixed strategies.

Theorem 5. Fundamental Theorem on Matrix Games.
In mixed strategies, every matrix game has at least one equilibrium point.

Again, the corresponding mixed strategies of the first and second player are called **maximin** and **minimax strategies**, respectively.

In other words, for every matrix A there exist vectors $\boldsymbol{p}^* \in S^s, \boldsymbol{q}^* \in T^s$, such that:

$$\boldsymbol{p}A\boldsymbol{q}^{*T} \leq \boldsymbol{p}^*A\boldsymbol{q}^{*T} \leq \boldsymbol{p}^*A\boldsymbol{q}^T \qquad \text{for all } \ \boldsymbol{p} \in S^s, \ \boldsymbol{q} \in T^s. \tag{4.18}$$

Theorem 6. *Equilibrium mixed strategies in a matrix game do not change when the same (but arbitrarily chosen, positive or negative) number c is added to all elements of the matrix. The value of the game with the matrix changed in this way, is $v + c$ where v is the value of the original game.*

4.5.3 Graphical Solution of $2 \times n$ Matrix Games

Expected values of player 1 for his mixed strategy $(p, 1-p)$ and second player's pure strategies are:

$$h_j(p) = pa_{1j} + (1-p)a_{2j}, \qquad j = 1, 2, \ldots, n. \tag{4.19}$$

For every $p \in \langle 0, 1\rangle$ the value $\min_{j=1,2,\ldots,n} g_j(p)$ gives the least guaranteed profit to player 1. Since the game is antagonistic, player 1 is looking for such value of p that maximizes this guaranteed profit:

$$p^* := \arg \max_{p\in\langle 0,1\rangle} \min_{j=1,2,\ldots,n} g_j(p). \tag{4.20}$$

First consider the function

$$\varphi(p) := \min_{j=1,2,\ldots,n} g_j(p). \tag{4.21}$$

This function is concave and it consists of a finite number of line segments; it is therefore easy to find it maximum. The sought-after value of the game is then

$$v = \varphi(p^*) := \max_{p\in\langle 0,1\rangle} \varphi(p) \tag{4.22}$$

and the sought-after mixed equilibrium strategy of player 1 is $(p^*, 1-p^*)$.

If the extreme occurs in a point p^* where $g_j(p^*) = g_k(p^*) = v$ for unique strategies j, k then the components of a mixed equilibrium strategy of player 2 with indices different from j, k are equal to zero. The components that can be non-zero can be obtained by solving one of the equation systems

$$a_{1j}q_j + a_{1k}q_k = v, \qquad q_j + q_k = 1, \quad q_j \geq 0, \quad q_k \geq 0, \tag{4.23}$$

$$\text{or} \qquad a_{2j}q_j + a_{2k}q_k = v, \qquad q_j + q_k = 1, \quad q_j \geq 0, \quad q_k \geq 0. \tag{4.24}$$

☛ ***Example 16.*** Graphical solution of a matrix game given by the matrix

$$M = \begin{pmatrix} 5 & 5/2 & 3 \\ 4 & 8 & 6 \end{pmatrix}.$$

$$\begin{aligned} g_1(p) &= 5p + 4(1-p) = p + 4 \\ g_2(p) &= \frac{5}{2}p + 8(1-p) = -\frac{11}{2}p + 8 \\ g_3(p) &= 3p + 6(1-p) = -3p + 6 \end{aligned}$$

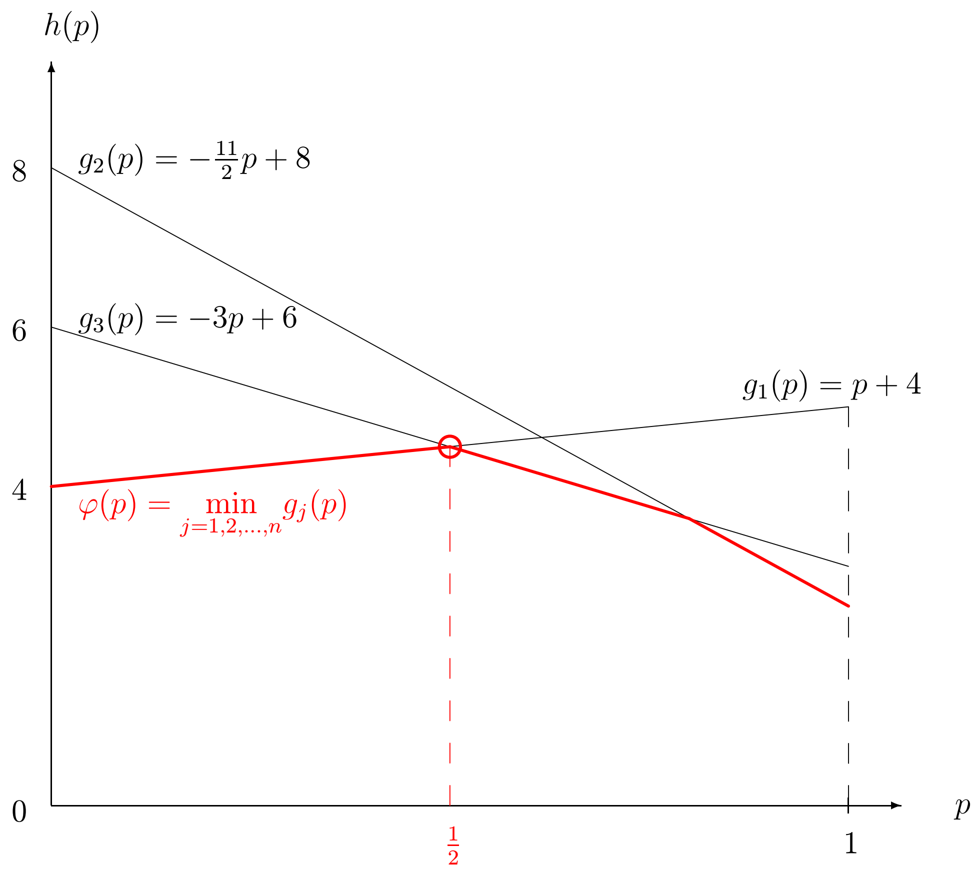

Fig. 4.7: Graphical Solution of the Matrix Game from Example 16

Function $\varphi(p)$ takes the maximal value for $p = \frac{1}{2}$ and this maximal value is

$$v(M) = 4.5.$$

Solving the equation system

$$5q_1 + 3q_3 = 4.5, \qquad q_1 + q_3 = 1, \quad q_1 \geq 0, \quad q_3 \geq 0,$$

we get $q_1 = 0.75$, $q_2 = 0.25$.

Hence the equilibrium pair is

$$\boldsymbol{p}^* = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \boldsymbol{q}^* = \left(\frac{3}{4}, \frac{1}{4}\right).$$

4.5.4 General Solution of Matrix Games – Linear Programming

Consider a matrix game given by a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix} \tag{4.25}$$

and mixed strategies

$$\begin{aligned} \boldsymbol{p} &= (p_1, p_2, \dots, p_m), \quad p_1 + p_2 + \dots + p_m = 1, \quad p_i \geq 0 \quad \forall i \in \{1, 2, \dots, m\}, \\ \boldsymbol{q} &= (q_1, q_2, \dots, q_n), \quad\quad q_1 + q_2 + \dots + q_n = 1, \quad q_j \geq 0 \quad \forall j \in \{1, 2, \dots, n\}. \end{aligned}$$

Suppose that all elements of the matrix A are positive (if not, it is possible to add a positive number, high enough – as we know, the new game would be strategically equivalent).

We will proceed analogously to searching pure equilibrium strategies.

For arbitrary but fixed $\boldsymbol{p}$, the first player searches his **minimal guaranteed payoff** h.

Consider

$$h = \min_{\forall j} \{a_{1j}p_1 + a_{2j}p_2 + \dots + a_{mj}p_m\} \,. \tag{4.26}$$

Obviously, we have

$$h \leq a_{1j}p_1 + a_{2j}p_2 + \dots + a_{mj}p_m \text{ for all } \; j \in \{1, 2, \dots, n\}. \tag{4.27}$$

For any j the expression on the right gives the expected payoff to the first player when he chooses a mixed strategy $\boldsymbol{p}$ and the second player chooses a pure strategy t_j . Expected value of the payoff $\pi(\boldsymbol{p}, \boldsymbol{q})$ for the mixed strategy $\boldsymbol{q}$ of the second player is a linear combination of these values with coefficients $q_1, q_2, \dots, q_n$, that sum up to 1. We can easily perceive that putting the mention linear combination on the right side of the inequality (4.27), remains unchanged:

$$\begin{array}{rcl} q_1\, h & \leq & q_1(a_{11}p_1 + a_{21}p_2 + \dots + a_{m1}p_m) \\ q_2\, h & \leq & q_2(a_{12}p_1 + a_{22}p_2 + \dots + a_{m2}p_m) \\ \dots & \dots & \dots \\ q_n\, h & \leq & q_n(a_{1n}p_1 + a_{2n}p_2 + \dots + a_{mn}p_m) \\ \hline \underbrace{(q_1 + q_2 + \dots + q_n)}_{1} h & \leq & \displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j = \pi(\boldsymbol{p}, \boldsymbol{q}) \\ h & \leq & \pi(\boldsymbol{p}, \boldsymbol{q}) \end{array}$$

The value h is therefore a minimal guaranteed payoff to the player 1, whichever pure or mixed strategy is chosen by the opponent (due to (4.26), h is the greatest number satisfying the last inequality).

Let us divide the inequalities (4.27) by h

$$1 \leq a_{1j}\frac{p_1}{h} + a_{2j}\frac{p_2}{h} + \cdots + a_{mj}\frac{p_m}{h}$$

and denote

$$y_i = \frac{p_i}{h}\ ; \qquad \text{obviously we have:} \quad y_1 + y_2 + \cdots + y_m = \frac{1}{h}\ .$$

We come to the inequality

$$1 \leq a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m\ . \tag{4.28}$$

To maximize the minimal guaranteed payoff means to maximize h, i.e.

Minimize

$$\frac{1}{h} = y_1 + y_2 + \cdots + y_m$$

under the conditions

$$1 \leq a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m\ , \qquad j = 1, 2, \ldots, n\ . \tag{4.29}$$

This is exactly the **dual problem of linear programming,** whose solution provides us the corresponding strategy $\boldsymbol{p}$.

Analogously for the second player. He is looking for h and $\boldsymbol{q}$ such that

$$h \geq a_{i1}q_1 + a_{i2}q_2 + \cdots + a_{in}q_n \quad \text{for all} \ \ i \in \{1, 2, \ldots, m\}, \tag{4.30}$$

with $q_1 + q_2 + \cdots + q_n = 1, q_j \geq 0$ for all $\forall j \in \{1, 2, \ldots, n\}$.

Let us divide the inequality (4.30) by h

$$1 \geq a_{i1}\frac{q_1}{h} + a_{i2}\frac{q_2}{h} + \cdots + a_{in}\frac{q_n}{h}$$

and denote

$$x_j = \frac{q_j}{h}\ ; \qquad \text{obviously we have:} \quad x_1 + x_2 + \cdots + x_n = \frac{1}{h}\ .$$

We come to the inequality

$$1 \geq a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m\ . \tag{4.31}$$

To minimize h therefore means:

maximize

$$\frac{1}{h} = x_1 + x_2 + \cdots + x_n$$

under the conditions

$$1 \geq a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m\ , \qquad i = 1, 2, \ldots, m\ . \tag{4.32}$$

This is exactly the **dual problem of linear programming** (if h should be the value of the game, it is necessary for both numbers to be the same).

4.6 GAMES AGAINST P-INTELLIGENT PLAYERS

4.6.1 Fundamental Concepts

Definition 11. A player behaving with probability p like a normatively intelligent player and with probability $1-p$ like a random mechanism will be called a ***p*-intelligent player**.

Parameter p characterizes the degree of deviation from the rational decision making. If $p = 0$, the player behaves in fact as a random mechanism, if $p = 1$, he is an intelligent player.

It is clear that it is not reasonable to apply the same strategies against the p-intelligent opponents as against intelligent opponents. Consider the case of matrix games.

Definition 12. The optimal strategy for the intelligent player is a row of the matrix A that maximizes the mean value of payoff when the p-intelligent player applies strategy

$$s(p) = py^* + (1-p)r$$

where y^* is a Nash equilibrium strategy of player 2 and r is a uniform probability distribution over columns.

☛ ***Example 17.*** Investigate a matrix game defined by the matrix

$$\begin{pmatrix} 3 & 3 & 3 & 3 \\ 7 & 1 & 7 & 7 \\ 3 & 1 & -1 & 2 \\ 8 & 0 & 8 & 8 \end{pmatrix}$$

Solution

The unique pair of equilibrium strategies is

$$x^* = (1, 0, 0, 0), \qquad y^* = (0, 1, 0, 0).$$

If player 2 is p-intelligent, player 1 expects that player 2 is going to use the strategy

$$s(p) = p(1, 0, 0, 0) + (1-p)(1/4, 1/4, 1/4, 1/4) = (1-p, 1+3p, 1-p, 1-p)/4\ .$$

We can easily verify that

- the first row is an optimal strategy for player 1 if $p \in \langle 5/9, 1\rangle$
- the second row is an optimal strategy for player 1 if $p \in \langle 1/3, 5/9\rangle$
- the fourth row is an optimal strategy for player 1 if $p \in \langle 0, 1/3\rangle$
- the third row is never an optimal strategy

How much we may lose if we apply a strategy which is optimal against a fully intelligent player when we play against a partly intelligent player?

Definition 13. The function $f(p)$, representing the average additional player's 1 profit due to his deviation from the equilibrium strategy x^* is called **excess function**.

If player 1 uses the optimal strategy against the p-intelligent player, he receives the payoff

$$\max_i a^{(i)} s(p),$$

where $a^{(i)}$ means the i-th row of the matrix A and i goes over all rows of A, that is $i = 1, 2, \ldots, m$. If he mechanically applies the strategy against the fully intelligent opponents, he receives

$$x^{*T} A s(p).$$

Therefore,

$$f(p) = \max_i \left[a^{(i)} s(p)\right] - x^{*T} A s(p). \tag{4.33}$$

☛ ***Example 18.*** For the game from example 17 we have

$$f(p) = \begin{cases} \frac{13}{4} - \frac{21}{4}p & \text{for} \quad p \in \langle 0, \frac{1}{3} \rangle \\ \frac{11}{4} - \frac{15}{4}p & \text{for} \quad p \in \langle \frac{1}{3}, \frac{5}{9} \rangle \\ 0 \quad \text{for} \quad p \in \langle \frac{5}{9}, 1 \rangle \end{cases}$$

The following theorem confirms the fact that taking into account the decreased intelligence of the opponent is always an advantage.

Theorem 7. *For any matrix game the excess function is a nonnegative, by parts linear, continuous and nonincreasing function on the interval* $\langle 0, 1 \rangle$.

In other words, the difference between what we get when applying the strategy based on the correct assessment of the opponent's intelligence is always at least so great as when we simply apply the strategy optimal against normatively intelligent player. The difference decreases or remains the same when the intelligence of the opponent increases.