5 TWO-PLAYER COOPERATIVE GAMES WITHOUT TRANSFERABLE PAYOFFS

5.1 FUNDAMENTAL CONCEPTS

In this part we will investigate the situations in which the players are allowed to make binding agreements about which strategies to play before the game starts, but they can not redistribute the payoff.

Definition 1. Let G be a two-player bimatrix game with $m \times n$ payoff matrices A, B. A **joint strategy** is an $m \times n$ probability matrix $P = (p_{ij})$. Thus

$$p_{ij} \geq 0 \quad \text{for} \quad 1 \leq i \leq m,\ 1 \leq j \leq n, \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij} = 1. \tag{5.1}$$

Thus, a joint strategy assigns a probability to each pair of pure strategies. The expected payoff to the row player due to the joint strategy P is

$$u(P) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}a_{ij}, \qquad v(P) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}b_{ij} \tag{5.2}$$

☛ ***Example 1.*** Consider the bimatrix game given by

$$\begin{pmatrix} (2,0) & (-1,1) & (0,3) \\ (-2,-1) & (3,-1) & (0,2) \end{pmatrix} \tag{5.3}$$

One of possible joint strategies is described by the matrix

$$\begin{pmatrix} 1/8 & 0 & 1/3 \\ 1/4 & 5/24 & 1/12 \end{pmatrix}$$

and it says that the pair of pure strategies (row 1, column 3) will be played with probability 1/3, the pair of pure strategies (row 2, column 3) will be played with probability 1/12, etc. The expected payoff for the first player is in this case

$$u(P) = \frac{1}{8}\cdot 2 + 0\cdot(-1) + \frac{1}{3}\cdot 0 + \frac{1}{4}\cdot(-2) + \frac{5}{24}\cdot 3 + \frac{1}{12}\cdot 0 = \frac{3}{8}\ .$$

Definition 2. Cooperative payoff region is the set

$$\mathbf{P} = \{(u(P), v(P)) : \ P \text{ is a joint strategy}\}. \tag{5.4}$$

From the definition we can directly deduce that the cooperative payoff region is always a **convex, closed and bounded set**, which contains the corresponding non-cooperative payoff region

$$\Pi = \{(u(\boldsymbol{p}, \boldsymbol{q}), v(\boldsymbol{p}, \boldsymbol{q})) : \ \boldsymbol{p}, \ \boldsymbol{q} \text{ are mixed strategies of player 1 and 2}\}. \tag{5.5}$$

In a **cooperative game** the players are allowed to make an agreement about which joint strategy to choose.

☛ ***Example 2.*** **The Battle of the Buddies** is a game given by the bimatrix

$$\begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix}. \tag{5.6}$$

The cooperative payoff region looks in this case like this:

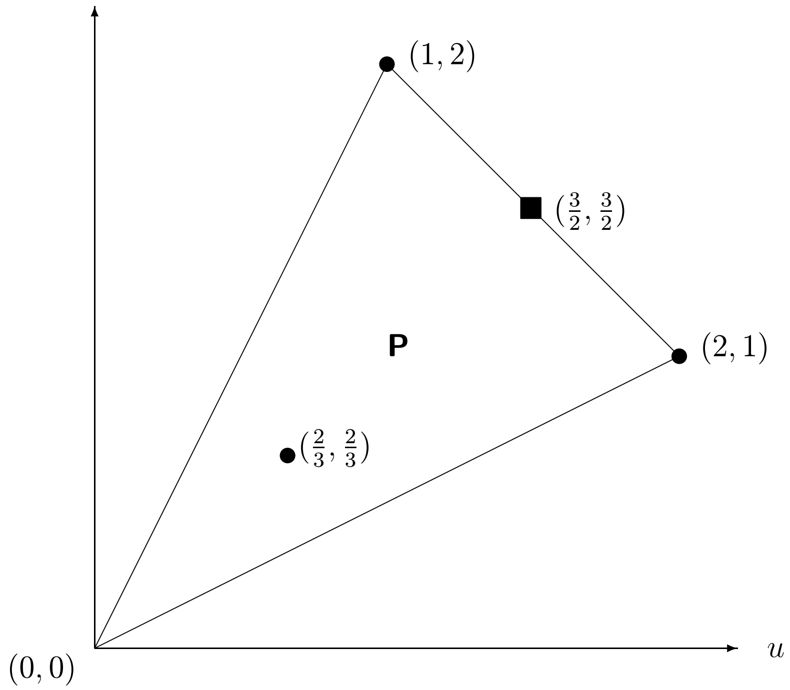

Fig. 5.1: Cooperative Payoff Region for the Battle of the Buddies

Definition 3. The payoff pair $(\hat{u}, \hat{v}) \in \mathbf{P}$ is called **Pareto optimal** or **non-dominated** if there does not exist any other payoff pair $(u, v) \in \mathbf{P}$ for which

$$u \geq \hat{u} \qquad \text{and} \qquad v \geq \hat{v},$$

withal at least one inequality is strict.

Intuitive ideas about which payoff pairs are the candidates for an agreement are summarized in the following definition:

Definition 4. The **bargaining set** for two-player cooperative game is the set of all *Pareto optimal* payoff pairs $(u, v) \in \mathbf{P}$ such that

$$u \geq v_1, \qquad v \geq v_2,$$

where v_1, v_2 are the maximin values, i.e.

$$v_1 = \max_{\boldsymbol{p}} \min_{\boldsymbol{q}} u(\boldsymbol{p}, \boldsymbol{q}), \qquad v_2 = \max_{\boldsymbol{q}} \min_{\boldsymbol{p}} v(\boldsymbol{p}, \boldsymbol{q})$$

(these values can be guaranteed by the players without a cooperation).

For the game from example 1, the bargaining set is represented in Fig. 5.2. Maximin values are $v_1 = 0$, $v_2 = 2$ in this case, the bargaining set is therefore the line segment between the points $(0, 3)$ and $(3/4, 2)$.

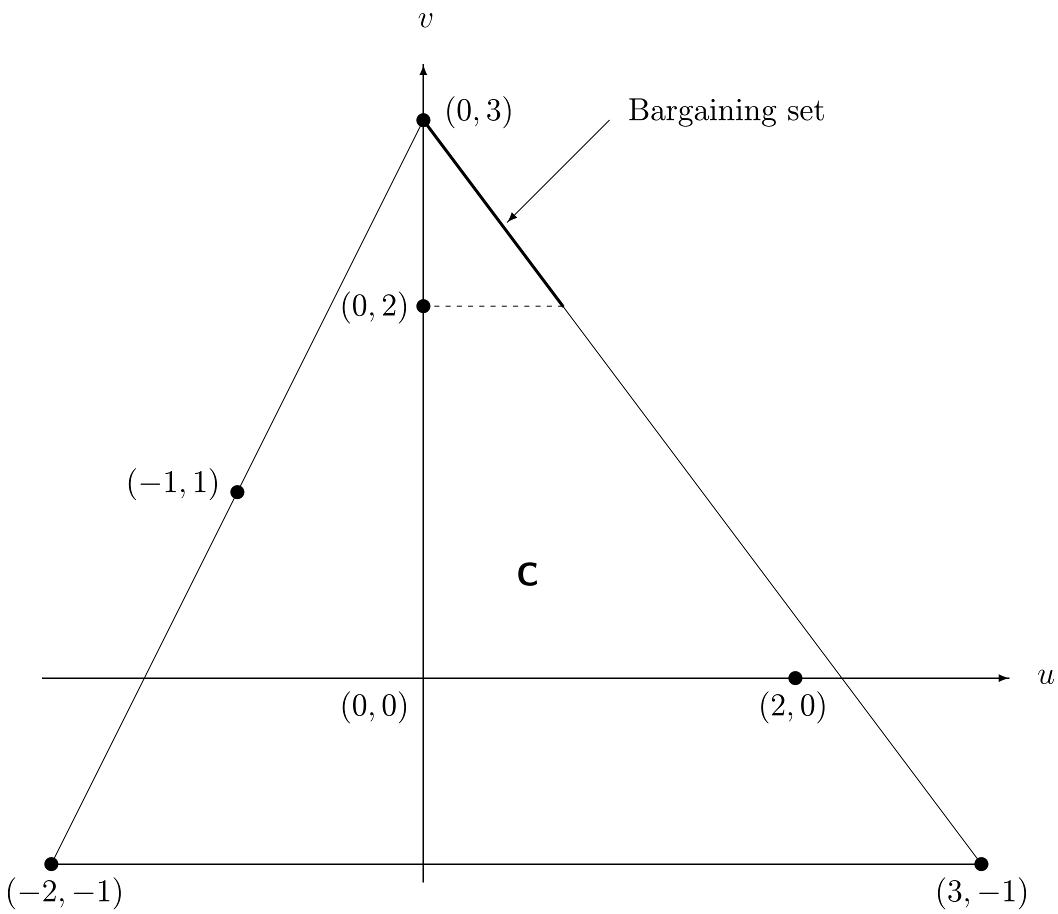

Fig. 5.2: Bargaining Set for the Game from Example 1

5.2 CONVEX SETS

Let us remind several concepts and properties concerning convex sets.

Definition 5. A set $\mathbf{S} \subset \mathbb{R}^n$ is called **convex** if, for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{S}$ and every real number t, $0 \leq t \leq 1$, it is:

$$t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathbf{M}$$

In other words, the set $\mathbf{S}$ is convex, if every line segment whose end-points are in $\mathbf{S}$ lies entirely in $\mathbf{S}$ (compare Fig. 5.3).

Definition 6. Let $\mathbf{F} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\}$ be a finite subset of $\mathbb{R}^n$. A **convex combination** of the set $\mathbf{F}$ is defined as a vector

$$\boldsymbol{w} = \sum_{i=1}^{k} t_i \boldsymbol{x}_i \qquad \text{where } t_1 \geq 0, \ldots, t_k \geq 0, \quad t_1 + \cdots + t_k = 1. \tag{5.7}$$

By induction, if a given set $\mathbf{S}$ is convex, then each convex combination of its points lies again in $\mathbf{S}$.

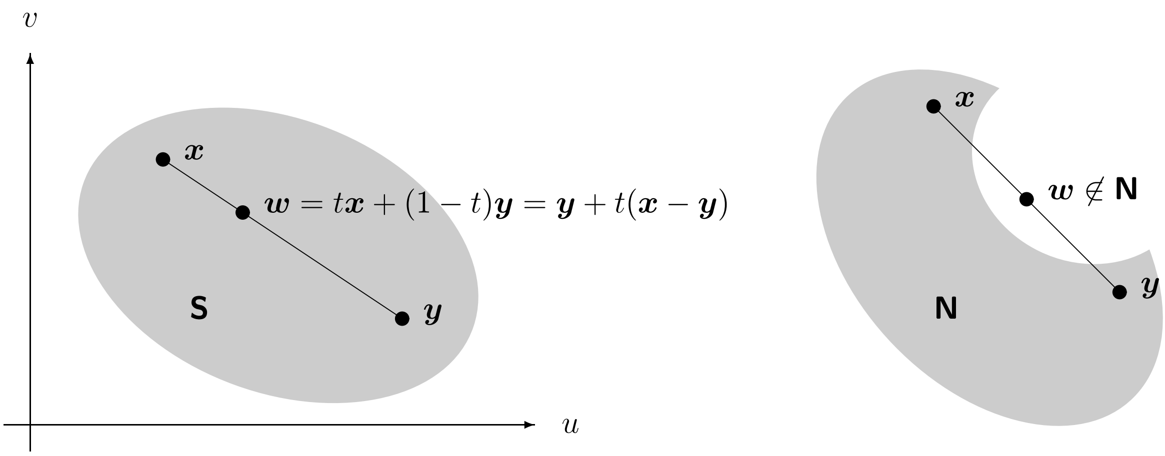

FIG. 5.3: CONVEX SET $\mathbf{S}$ AND NONCONVEX SET $\mathbf{N}$

Definition 7. Let $\mathbf{A}$ be a subset of $\mathbb{R}^n$. **Convex hull** of the set $\mathbf{A}$ is defined as the set of all convex combinations of finite subsets of the set $\mathbf{A}$. For the convex hull we will use the symbol $\operatorname{conv}(\mathbf{A})$.

Prove that the convex hull is a convex set and every convex set containing $\mathbf{A}$ contains $\operatorname{conv}(\mathbf{A})$, too.

In Figure 5.1, the convex hull of points $(0,0)$, $(1,2)$ a $(2,1)$ is a triangle $\mathbf{C}$ with vertices in these points; the same triangle is a convex hull of the set which in addition to the above points contains also $(\frac{2}{3},\frac{2}{3})$ and $(\frac{3}{2},\frac{3}{2})$ or possibly other points lying inside or at the boundary of the considered triangle. In Figure 5.2 the convex hull of all depicted points in the triangle $\mathbf{C}$ with vertices in points $(2,-1)$, $(3,-1)$ and $(0,3)$. In the left part of Figure 5.3 the ellipse $\mathbf{S}$ is a convex hull of the points forming its boundary.

Since the coefficients t_i in the convex combination (5.7) have the properties of probabilities, we receive the following theorem:

Theorem 1. *Let G be a two-player game given by an $m \times n$ bimatrix C. Cooperative payoff region is the convex hull of the set of points in $\mathbb{R}^2$ whose coordinates are the elements of the bimatrix C.*

Proof. If P is a joint strategy, then the corresponding payoff pairs are

$$(u(P), v(P)) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} c_{ij}.$$

All these points form a convex hull of the set

$$\{c_{ij}, 1 \le i \le m, 1 \le j \le n\}.$$

Conversely, any point of the convex hull of this set is a payoff pair. □

Definition 8. The set $\mathbf{S} \subset \mathbb{R}^2$ is called **symmetric** if, for every $u, v \in \mathbb{R}$ it is:

$$(v,u) \in \mathbf{S} \iff (u,v) \in \mathbf{S}.$$

Definition 9. Consider a set $\mathbf{A} \subset \mathbb{R}^2$. The **symmetric convex hull** of $\mathbf{A}$ is defined to be the convex hull of the set

$$\mathbf{A} \cup \{(v,u) : \ (u,v) \in \mathbf{A}\}.$$

It is denoted $\mathrm{sconv}(\mathbf{A})$.

Proposition 1. *Let $\mathbf{A} \subset \mathbb{R}^2$ and k is such a number that for every point $(u,v) \in \mathbf{A}$ we have:*

$$u + v \le k.$$

Then the same inequality holds for every point of the symmetric convex hull $\mathrm{sconv}(\mathbf{A})$.

5.3 BARGAINING

5.3.1 Nash Bargaining Axioms

The following theory was developed by John Nash in 1950 and it represents an attempt to state a fair method how to decide which payoff pair in the bargaining set should be chosen. The fundamental idea lies in the derivation of a so-called **arbitration procedure** Ψ, which for the set $\mathbf{P}$ and the „status quo“ point $(u_0, v_0) \in \mathbf{P}$ provides the payoff pair – so-called **arbitration point**, which is even-handed to both players. As the „status quo“ point, the pair of maximin strategies is usually considered.

We can imagine the arbitration procedure Ψ as a process in which an independent person – an arbiter is called to solve the conflict.

The arbitration procedure is demanded to fulfill the following properties that can be understood as fairness and consistency principles that can lead the arbiter in his decision.

Definition 10 – NASH BARGAINING AXIOMS.

For the payoff region $\mathbf{P}$ and the „status quo“ point $(u_0, v_0) \in \mathbf{P}$ denote

$$\Psi(\mathbf{P}, (u_0, v_0)) = (u^*, v^*).$$

1. **Individual Rationality:** $u^* \geq u_0, \;\; v^* \geq v_0$.
2. **Pareto Optimality:** the pair (u^*, v^*) is Pareto optimal.
3. **Feasibility:** $(u^*, v^*) \in \mathbf{P}$.
4. **Independence of Irrelevant Alternatives:**
   If $\mathbf{P}'$ is a payoff region contained in $\mathbf{P}$ and both pairs $(u_0, v_0), \; (u^*, v^*) \in \mathbf{P}'$, then
   $$\Psi(\mathbf{P}', (u_0, v_0)) = (u^*, v^*).$$
5. **Independence Under Linear Transformations:**
   Suppose $\mathbf{P}'$ is obtained from $\mathbf{P}$ by the linear transformation
   $$u' = au + b, \qquad v' = cv + d, \qquad \text{where} \quad a, c > 0,$$
   then
   $$\Psi(\mathbf{P}', (au_0 + b, cv_0 + d)) = (au^* + b, cv^* + d).$$
6. **Symmetry:** Suppose that $\mathbf{P}$ is symmetric (that is, $(u, v) \in \mathbf{P} \Leftrightarrow (v, u) \in \mathbf{P}$) and $(u_0, v_0) \in \mathbf{P}$, then $u^* = v^*$.

Theorem 2. *There exists a unique arbitration procedure Ψ satisfying Nash's axioms.*

Proof.

- **Construction of Ψ.**

Case (i) There exists $(u,v) \in \mathbf{P}$ such that $u > u_0$ and $v > v_0$.

Let $\mathbf{K}$ be the set of all such points (u,v). Define

$$g(u,v) = (u-u_0)(v-v_0), \qquad \text{for} \quad (u,v) \in \mathbf{K}.$$

It can be proved that there exists a unique point (u^*,v^*) in which the function $g(u,v)$ attains its maximum value. Define

$$\Psi(\mathbf{P},(u_0,v_0)) = (u^*,v^*).$$

Case (ii) No $(u,v) \in \mathbf{P}$ exists such that $u > u_0$ and $v > v_0$.
Consider the following three subcases:

Case (iia) There exists $(u_0,v) \in \mathbf{P}$ such that $v > v_0$.

The largest v with this property for which it is $(u_0,v) \in \mathbf{P}$ denote v^*.
Define

$$\Psi(\mathbf{P},(u_0,v_0)) = (u_0,v^*).$$

Case (iib) There exists $(u,v_0) \in \mathbf{P}$ such that $u > u_0$.

The largest u with this property for which it is $(u,v_0) \in \mathbf{P}$ denote u^*.
Define

$$\Psi(\mathbf{P},(u_0,v_0)) = (u^*,v_0).$$

Case (iic) Neither (iia) nor (iib) is true.

Define

$$\Psi(\mathbf{P},(u_0,v_0)) = (u_0,v_0).$$

Cases (iia) and (iib) can not both be true: suppose that they are and define

$$(u',v') = \frac{1}{2}(u_0,v) + \frac{1}{2}(u,v_0).$$

Then (u',v') is in $\mathbf{P}$ (from convexity) and satisfies the condition of Case (i) – since Case (i) does not hold, this is a contradiction.

- **Verifying Nash Axioms.**

Axioms (1) and (3) apparently hold in all cases.
Axiom (2): If it does not hold than a point $(u,v) \in \mathbf{P}$ would exist that would dominate the point (u^*,v^*) and that would be different from it.
In Case **(i)** it would be

$$(u-u_0) \geq (u^*-u_0), \qquad (v-v_0) \geq (v^*-v_0)$$

and at least one of these inequalities would be strict (since $(u,v) \neq (u^*,v^*)$). Thus,

$$g(u,v) > g(u^*,v^*),$$

which is a **contradiction** to the construction of (u^*,v^*).

In Case **(iia)** it must be $u^* = u_0 = u$ because (iib) does not hold. It is therefore $v > v^*$, which is a contradiction to the definition of v^*. In Case **(iib)** we can proceed similarly. In Case **(iic)** we have $(u^*, v^*) = (u_0, v_0)$; if it would be $u > u_0$, then Case (iib) would hold, for $v > v_0$ it would be the Case (iia), which is again a contradiction.

Axiom (4): In case **(i)** the maximum value of the function g subject to the constrained $\mathbf{P} \cap \mathbf{P}'$, is less of equal to its maximum value on the set $\mathbf{S}$. Since $(u^*, v^*) \in \mathbf{P}'$, these maximums are equal. Thus,

$$\Psi(\mathbf{P}', (u_0, v_0)) = \Psi(\mathbf{P}, (u_0, v_0)).$$

Similarly for other cases.

Axiom (5): In case **(i)**, the same case holds also for a payoff region $\mathbf{P}'$ with the status quo point $(au_0 + b, cv_0 + d)$. Thus,

$$(u' - (au_0 + b))(v' - (cv_0 + d)) = ac(u - u_0)(v - v_0).$$

Since $a, c > 0$, the function on the left side attains its maximum at $(au^* + b, cv^* + d)$. In case (i) the axiom (5) therefore holds. The process in other cases is similar.

Axiom (6): If $u^* \neq v^*$ than from the symmetry $(v^*, u^*) \in \mathbf{P}$; in case **(i)** it would be

$$g(v^*, u^*) = g(u^*, v^*).$$

According to proposition 2 the function g attains its maximum at a unique point, what is a contradiction. Due to symmetry, cases **(iia)** and **(iib)** can not occur.

- **Uniqueness.**

The proof is done by a contradiction resulting from the assumption that there exists another arbitration procedure $\overline{\Psi}$ satisfying Nash axioms. Since these procedures are different, there exists a payoff region $\mathbf{P}$ and „status quo“ point $(u_0, v_0) \in \mathbf{P}$, for which

$$(\overline{u}, \overline{v}) = \overline{\Psi}(\mathbf{P}, (u_0, v_0)) \neq \Psi(\mathbf{P}, (u_0, v_0)) = (u^*, v^*).$$

Proposition 2. *Let* $\mathbf{P}$ *be a payoff region and* $(u_0, v_0) \in \mathbf{P}$. *Assume that there exists a point* $(u, v) \in \mathbf{P}$ *with*

$$u > u_0, \quad v > v_0;$$

Denote with $\mathbf{C}$ *the set of all points* (u, v) *with the mentioned property. Define on* $\mathbf{C}$ *a function*

$$g(u, v) = (u - u_0)(v - v_0).$$

Then g *attains its maximum on* $\mathbf{C}$ *at one and only one point.*

5.3.2 Examples

In the Battle of the Buddies studied in example 2, the payoff region is symmetric, the pair of maximin values is $(\frac{2}{3}, \frac{2}{3})$. As "status quo" point consider $(u_0, v_0) = (\frac{2}{3}, \frac{2}{3})$. Due to axiom (6) the arbitration point must have a form of (a, a). Since due to axiom (2) the point (a, a) must be Pareto optimal, it must be $a = \frac{3}{2}$ (see Fig. 5.1). It is also a point to which an intuition would lead us.

☛ ***Example 3.*** Consider a two-player cooperative game with the bimatrix

$$\begin{pmatrix} (2,-1) & (-2,1) & (1,1) \\ (-1,2) & (0,2) & (1,-2) \end{pmatrix}. \qquad (5.8)$$

Maximin values are $v_1 = -\frac{2}{5}, v_2 = 1$ (verify!).

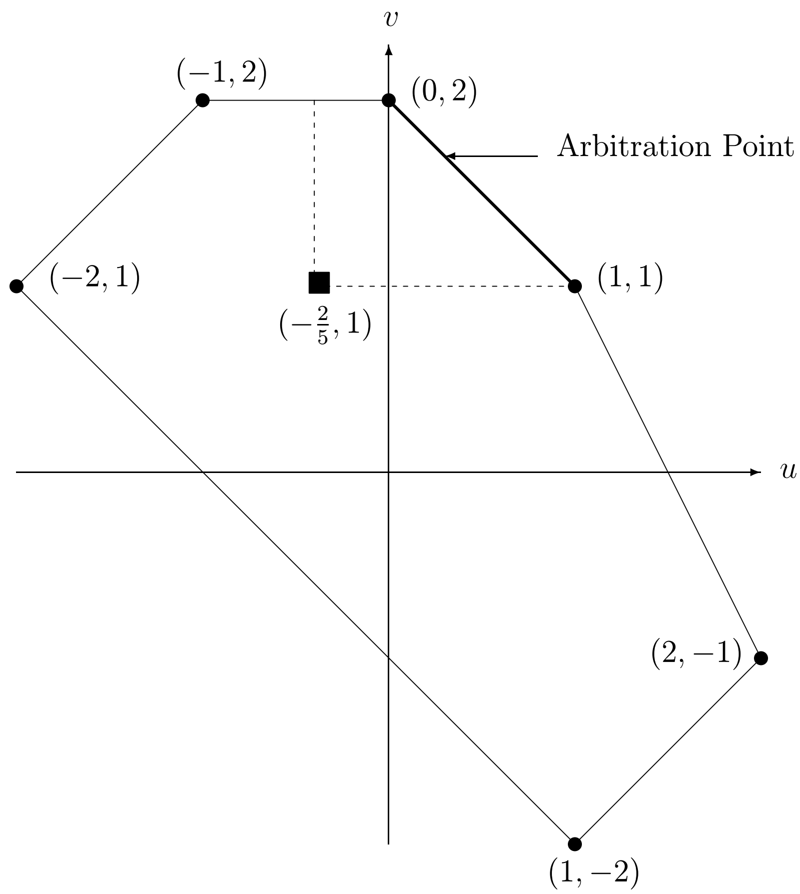

FIG. 5.4: PAYOFF REGION AND ARBITRATION PROCEDURE FOR THE GAME FROM EXAMPLE 3.

Set $(u_0, v_0) = (-\frac{2}{5}, 1)$. Arbitration point must be found among the points of the payoff function that dominate the point $(-\frac{2}{5}, 1)$ but that are not dominated by any other points – i.e. in this case on the line-segment with end-points $(0, 2)$ and $(1, 1)$ which represents the bargaining set. According to the construction of the arbitration point we search the maximum of the function

$$g(u,v) = (u-u_0)(v-v_0) = \left(u+\frac{2}{5}\right)(v-1)$$

in the line segment given by an equality $v = -u+2$. Thus, the only problem is to find the extreme of the function of one unknown.

$$g(u,-u+2) = \left(u+\frac{2}{5}\right)(-u+1) = -u^2+\frac{3}{5}u+\frac{2}{5}\ .$$

With help of a simple calculus we obtain

$$u = \frac{3}{10}\ , \quad v = \frac{17}{10}\ .$$

☛ ***Example 4.*** Consider a cooperative game given by the bimatrix

$$\begin{pmatrix} (5,1) & (7,4) & (1,10) \\ (1,1) & (9,-2) & (5,1) \end{pmatrix}. \tag{5.9}$$

Maximin values are $v_1 = 3, v_2 = 1$. Bargaining set consists of two line segments; apply the process from example 3 for both segments. For one line segment, maximum lies out of it and for the second one we obtain an arbitration point

$$u = \frac{13}{2}\ , \quad v = \frac{9}{2}\ .$$

☛ ***Example 5.*** Consider cooperative game given by the bimatrix

$$\begin{pmatrix} (2,-3) & (-1,3) \\ (0,1) & (1,-2) \end{pmatrix}. \tag{5.10}$$

Maximin values are $v_1 = \frac{1}{2},\ v_2 = -\frac{1}{3}$. Bargaining set is formed by the line segment with end-points $(\frac{1}{2},0)$ and $(\frac{2}{3},-\frac{1}{3})$. Maximum of the function

$$g(u,v) = (u-\frac{1}{2})(v+\frac{1}{3})$$

in the line segment

$$v = -2u+1, \qquad \frac{1}{2} \le u \le \frac{2}{3}$$

occurs in $(\frac{7}{12},-\frac{1}{6})$; this is the searched arbitration point.