6 N-PLAYER COOPERATIVE GAMES

6.1 GAMES IN CHARACTERISTIC FUNCTION FORM

6.1.1 Fundamental Concepts

In the previous chapter the players were allowed to coordinate their strategies but they could not share their payoffs. In games studied in this part full cooperation including payoff share is possible. We will assume that the agreements are completely binding.

Definition 1. Consider an N-player game; denote with Q the set of all players. **Coalition** is a group of players cooperating in strategy choices and payoff redistribution. **Coalition structure** is defined to be the set of all coalitions that can be formed by the players in a given situation. Coalitions will be denoted with K, L, Q, etc., or directly in the set form, for example $\{1\}$, $\{2, 3, 5\}$, etc.

Counter-coalition to a coalition $K \subseteq Q$ is the set of players

$$K^c = Q \setminus K = \{i \in Q; i \notin K\}.$$

The set of all players Q is called **grand coalition**, its counter-coalition, that is an empty set, is called an **empty coalition**.

In general, in total 2^N coalitions can form in an N-player game – this is precisely the number of subsets of the set Q.

Definition 2. A **Game in Characteristic Function Form** consists of the set of players

$$Q = \{1, 2, \ldots, N\}$$

and a real function v defined on the set of all coalitions, satisfying the following conditions:

$$v(\emptyset) = 0$$

and for every two disjoint coalitions K, L **superaditivity** holds:

$$v(K \cup L) \geq v(K) + v(L).$$

For the sake of simplicity, v will also denote the corresponding characteristic function form game itself.

Values of the characteristic function give the power of particular coalitions.

Definition 3. A game in characteristic function form is called **inessential** if

$$v(Q) = \sum_{i=1}^{N} v(\{i\}).$$

A game which is not inessential is called **essential**.

Theorem 1. *Let K be a coalition in an inessential game. Then*

$$v(K) = \sum_{i \in K} v(\{i\})$$

6.1.2 Imputations

Definition 4. Let v be a game in characteristic function form with the set of players

$$Q = \{1, 2, \dots, N\}.$$

An N-tuple $\boldsymbol{a}$ of real numbers is called an **imputation** if the following conditions hold:

- **Individual Rationality**: for all players i it is

$$a_i \geq v(\{i\}). \tag{6.1}$$

- **Collective Rationality**: We have

$$\sum_{i=1}^{N} a_i = v(Q). \tag{6.2}$$

Motivation – Individual Rationality:

If for some i it would be $a_i < v(\{i\})$ then a coalition that would bring only a_i to the player i never forms – for the player i it is more profitable to stay alone.

Collective Rationality:

It is:

$$\sum_{i=1}^{N} a_i \geq v(Q). \tag{6.3}$$

In the opposite case it would be

$$\beta = v(Q) - \sum_{i=1}^{N} a_i > 0.$$

For players it would be more advantageous to form a grand coalition and share the total profit $v(Q)$ so that each receives more – for example:

$$a_i' = a_i + \beta/N.$$

On the other side, it must also be

$$\sum_{i=1}^{N} a_i \leq v(Q). \tag{6.4}$$

Imagine some division $\boldsymbol{a}$ was realized, i.e. the players of a certain coalition K and the members of the corresponding counter-coalition K^c have agreed with such a division. Due to superaditivity we have:

$$\sum_{i=1}^{N} a_i = \sum_{i \in K} a_i + \sum_{i \in K^c} a_i = v(K) + v(K^c) \leq v(Q).$$

The conditions (6.3) and (6.4) give together the collective rationality condition (6.2).

Theorem 2. *Let v be a game in characteristic function form. If v is inessential, than it has a unique imputation, namely*

$$\boldsymbol{a} = (v(\{1\}), v(\{2\}), \ldots, v(\{N\})).$$

If v is essential, than it has infinitely many imputations.

Proof. For an inessential game v : If, for some j it would be

$$a_j > v(\{j\}),$$

then

$$\sum_{i=1}^{N} a_i > \sum_{i=1}^{N} v(\{i\}) = v(Q),$$

which is a contradiction to the collective rationality.

For an essential game v consider

$$\beta = v(Q) - \sum_{i=1}^{N} a_i > 0.$$

For any N-tuple $\boldsymbol{\alpha}$ of non-negative numbers whose sum is β, the relation

$$\boldsymbol{a}_i = v(\{i\}) + \alpha_i$$

defines an imputation. Since there exist infinitely many such numbers α, there exist an infinite number of imputations. □

Formalization of preferences:

Definition 5. Let v be a game in characteristic function form, K is a coalition, $\boldsymbol{a}, \boldsymbol{b}$ imputations. We say that the imputation $\boldsymbol{a}$ **dominates the imputation $\boldsymbol{b}$ through a coalition $\boldsymbol{K}$**, if the following conditions hold:

- $a_i > b_i$ for all $i \in K$,
- $\sum_{i \in K} a_i \leq v(K)$.

Dominance relation will be denoted by $\boldsymbol{a} \succ_K \boldsymbol{b}$.

The second condition says that $\boldsymbol{a}$ is **feasible**, that is, that the players in S can attain enough payoff so that a_i can actually be paid out to each player in the coalition K.

6.2 SOLUTION CONCEPTS

6.2.1 The Core

Intuitively it is clear that if an imputation is dominated by another one through some coalition, than the players of this coalition will try to break down the original coalition and settle the more advantagous one.

Definition 6. Let v be a game in characteristic function form. **The Core** of the game consists of all imputations that are not dominated by any other imputation through any coalition.

If an imputation $\boldsymbol{a}$ is in the core, then no group of players has a reason to form another coalition and replace $\boldsymbol{a}$ by another imputation.

The following theorem makes the decision whether an imputation is in the core easier:

Theorem 3. *Let v be a game in characteristic function form with N players, let $\boldsymbol{a}$ be an imputation. Then $\boldsymbol{a}$ is in the core of v if and only if*

$$\sum_{i \in K} a_i \geq v(K) \tag{6.5}$$

for every coalition K.

Proof. $\Rightarrow$ Suppose that for every coalition the relation (6.5) holds. If some other imputation $\boldsymbol{b}$ dominates $\boldsymbol{a}$ for some coalition K, then

$$\sum_{i \in K} b_i > \sum_{i \in K} a_i \geq v(K),$$

which violates the feasibility condition in the dominance definition. Thus, $\boldsymbol{a}$ must be in the core.

$\Leftarrow$ Suppose that $\boldsymbol{a}$ is in the core and K is a coalition for which

$$\sum_{i \in K} a_i < v(K).$$

We need to come to a contradiction. First notice that $K \neq Q$, since otherwise the collective rationality condition would not hold.

Further, there exists a player $j \in K^c$ for whom

$$a_j > v(\{j\}).$$

If not, than with respect to superaditivity:

$$\sum_{i=1}^{N} a_i < v(K) + \sum_{i \in K^c} a_i \leq v(Q),$$

which again violates the feasibility condition. We can therefore choose such $j \in K^c$ that there exists a number α for which

$$0 < \alpha \leq a_j - v(\{j\}) \qquad \text{and} \qquad \alpha \leq v(Q) - \sum_{i \in K} a_i.$$

If k denotes the number of players in the coalition K, we can define a new imputation $\boldsymbol{b}$ dominating $\boldsymbol{a}$ by

$$\begin{aligned} &b_i = a_i + \alpha/k \qquad \text{for } P_i \in K, \\ &b_j = a_j - \alpha, \\ &b_i = a_i \qquad \text{for all other } i. \end{aligned}$$

Such imputation $\boldsymbol{b}$ dominates the imputation $\boldsymbol{a}$ for K, which is a contradiction with the assumption that $\boldsymbol{a}$ is in the core. $\square$

Proposition 1. *Let v be a game in characteristic function form with N players and let $\boldsymbol{a}$ be an N-tuple of numbers. Then $\boldsymbol{a}$ is an imputation in the core if and only if it is*

- $\displaystyle\sum_{i=1}^{N} a_i = v(Q),$

- $\displaystyle\sum_{i \in K} a_i \geq v(K)$ *for every coalition K.*

Proof. Obviously, every imputation in the core satisfies both conditions.

Conversely, if an N-tuple $\boldsymbol{a}$ satisfies these conditions, then applying the second condition to one-player coalitions shows that individual rationality holds. The first condition is collective rationality, and thus $\boldsymbol{a}$ is in the core, by the theorem. $\square$

☛ ***Example 1.*** Consider a three-player game given by the table:

Strategy Triplets	Payoff Vectors
(1,1,1)	(-2,1,2)
(1,1,2)	(1,1,-1)
(1,2,1)	(0,1,-1)
(1,2,2)	(-1,2,0)
(2,1,1)	(1,-1,1)
(2,1,2)	(0,0,1)
(2,2,1)	(1,0,0)
(2,2,2)	(1,2,-2)

The set of players is $Q = \{1, 2, 3\}$, all possible coalitions are

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} = Q.$$

Consider a coalition $K = \{1, 3\}$. Counter-coalition is $K^c = \{2\}$. Coalition K has four joint strategies: $(1, 1), (1, 2), (2, 1), (2, 2)$. Counter-coalition has two pure strategies: 1, 2. When we are interested in what the coalition K is able to guarantee for itself, we consider a bimatrix game:

		Counter-coalition K^c	
	Strategy	1	2
	$(1, 1)$	$(0, 1)$	$(2, -1)$
Coalition K	$(1, 2)$	$(0, 1)$	$(-1, 2)$
	$(2, 1)$	$(2, -1)$	$(1, 0)$
	$(2, 2)$	$(1, 0)$	$(-1, 2)$

Maximin values of payoff functions are $3/4$ and $-1/3$, charakteristic function is therefore

$$v(\{1, 3\}) = 3/4, \quad v(\{2\}) = -1/3.$$

Similarly we obtain:

$$v(\{1, 2\}) = 1, \quad v(\{3\}) = 0 \quad v(\{2, 3\}) = 3/4, \quad v(\{1\}) = 1/4,$$

$$v(Q) = 1, \quad v(\emptyset) = 0.$$

The function v defined in this way is really a characteristic function – verify the superaditivity condition.

Imputations:

$$a_1 + a_2 + a_3 = 1, \qquad a_1 \geq 1/4, \quad a_2 \geq -1/3, \quad a_3 \geq 0.$$

For example:

$$(1/3, 1/3, 1/3), \quad (1/4, 3/8, 3/8), \quad (1, 0, 0).$$

The core:

$$\begin{aligned} a_1 + a_2 + a_3 &= 1 \\ a_1 &\geq 1/4 \\ a_2 &\geq -1/3 \\ a_3 &\geq 0 \\ a_1 + a_2 &\geq 1 \\ a_1 + a_3 &\geq 4/3 \\ a_2 + a_3 &\geq 3/4 \end{aligned}$$

From the first, fourth and fifth relation we have: $a_3 = 0$, $a_1 + a_2 = 1$. But it is also $a_1 \geq 4/3$, $a_2 \geq 3/4$. Thus the core is empty.

☛ ***Example 2.*** Consider a three-player game with characteristic function:

$$\begin{aligned} v(\{1\}) &= -1/2 \\ v(\{2\}) &= 0 \\ v(\{3\}) &= -1/2 \\ v(\{1,2\}) &= 1/4 \\ v(\{1,3\}) &= 0 \\ v(\{2,3\}) &= 1/2 \\ v(\{1,2,3\}) &= 1 \end{aligned}$$

The core:

$$\begin{aligned} a_1 + a_2 + a_3 &= 1 \\ a_1 &\geq -1/2 \\ a_2 &\geq 0 \\ a_3 &\geq -1/2 \\ a_1 + a_2 &\geq 1/4 \\ a_1 + a_3 &\geq 0 \\ a_2 + a_3 &\geq 1/2 \end{aligned}$$

This system has infinitely many solutions; for examle, the core contains the triplet $(1/3, 1/3, 1/3)$.

☛ ***Example 3.*** **Used Car Game.**

David has an old car. He no longer drives it and it is worth nothing to him unless he can sell it. Two people are interested in buying it: Mary and Frank. Mary values the car at 500 EUR, Frank at 700 EUR. The game consists of each of the prospective buyers bidding on the car, and David either accepting one of the bids, or rejecting both of them.

The core: (a_D, a_M, a_F); $\quad 500 \leq a_D \leq 700, \quad a_F = 700 - a_D, \quad a_M = 0.$

6.2.2 Shapley Value

Shapley value takes into account player's contribution to the success of the coalition to which he belongs.

Let v be a game in characteristic function form with N players, K a coalition consisting of k members, $i \in K$. The number

$$\delta(i, K) = v(K) - v(K \setminus \{i\})$$

is a measure of the value which the player i contributes to the coalition K when he joins it.

Coalition $K \setminus \{i\}$ has $k-1$ members and can be therefore created in

$$\binom{N-1}{k-1} = \frac{(N-1)!}{(k-1)!(N-k)!}$$

ways (the player i is out of the selection, he joins the coalition as the last one).

The mean value of player i's contribution to all k-player coalitions is

$$\begin{aligned} h_i(k) &= \sum_{K \subset Q,\ k=|K|} \frac{v(K) - v(K \setminus \{i\})}{\binom{N-1}{k-1}} = \\ &= \sum_{K \subset Q,\ k=|K|} \frac{(k-1)!(N-k)!}{(N-1)!} \left(v(K) - v(K \setminus \{i\})\right). \end{aligned} \tag{6.6}$$

The mean value of player i's contribution to all one-player, two-player, ..., N-player coalitions is given by

$$H_i = \sum_{k=1}^{N} \frac{h_i(k)}{N} = \sum_{K \subset Q,\ i \in K} \frac{(N-k)!(k-1)!}{N!} \left(v(K) - v(K \setminus \{i\})\right) \tag{6.7}$$

Definition 7. Shapley Vector of the N-player game in characteristic function form is defined to be the vector

$$\boldsymbol{H} = (H_1, H_2, \ldots, H_N), \tag{6.8}$$

whose i-th component H_i is given by (6.7).

The component H_i is called the **Shapley value** for the player i.

Theorem 4. *Let v be a game in characteristic function form. Than Shapley vector is an imputation.*

From the definition it is clear that Shapley vector always exists and is unique for a given game.

☛ ***Example 4.*** Find Shapley values of the game with characteristic function

$$v(Q) = 0, \quad v(\emptyset) = 0,$$

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = -1,$$

$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 1.$$

In this case we have

$$h_1(1) = -1, \qquad h_1(2) = \frac{2+2}{2} = 2, \qquad h_1(3) = -1,$$

Shapley value for each player is

$$H_i = \frac{-1+2-1}{3} = 0 \qquad \text{for } \ i = 1, 2, 3,$$

and Shapley vector is $\boldsymbol{H} = (0, 0, 0)$.

☛ ***Example 5.*** Consider a game with characteristic function

$$v(Q) = 200, \quad v(\emptyset) = 0,$$

$$v(\{1\}) = 100, \quad v(\{2\}) = 10 \quad v(\{3\}) = 0,$$

$$v(\{1,2\}) = 150, \quad v(\{1,3\}) = 110, \quad v(\{2,3\}) = 20.$$

In this case we have

$$h_1(1) = 100, \qquad h_2(1) = 10, \qquad h_3(1) = 0,$$

$$h_1(2) = \frac{140+110}{2}, \qquad h_2(2) = \frac{50+20}{2}, \qquad h_3(2) = \frac{10+10}{2},$$

$$h_1(3) = 180, \qquad h_2(3) = 90, \qquad h_3(3) = 50.$$

On the whole:

$$H_1(1) = \frac{100+125+180}{3} = 135,$$

$$H_1(2) = \frac{10+35+90}{3} = 45,$$

$$H_1(3) = \frac{0+10+50}{3} = 20,$$

Shapley vector: $\boldsymbol{H} = (135, 45, 20)$.

☞ ***Example 6.*** Consider the game from example 1 whose characteristic function is given by

$$v(Q) = 1, \quad v(\emptyset) = 0,$$

$$v(\{1\}) = \frac{1}{4}, \quad v(\{2\}) = -\frac{1}{3}, \quad v(\{3\}) = 0,$$

$$v(\{1,2\}) = 1, \quad v(\{1,3\}) = \frac{4}{3}, \quad v(\{2,3\}) = \frac{3}{4}.$$

In this case, Shapley values are

$$H_1 = \frac{2!0!}{3!} \cdot \frac{1}{4} + \frac{1!1!}{3!} \cdot \frac{4}{3} + \frac{1!1!}{3!} \cdot \frac{4}{3} + \frac{0!2!}{3!} \cdot \frac{1}{4} = \frac{11}{18},$$

similarly

$$H_2 = \frac{1}{36}, \qquad H_3 = \frac{13}{36}.$$

Shapley vector is therefore

$$\boldsymbol{H} = \left(\frac{11}{18}, \frac{1}{36}, \frac{13}{36}\right).$$

☞ ***Example 7.*** For the game from example 3 Shapley values are the following:

$$H_D = 43\,333,\overline{3}; \qquad H_M = 8\,333,\overline{3}; H_F \qquad 18\,333,\overline{3};$$

that is

$$\boldsymbol{H} = \left(43\,333,\overline{3};\ 8\,333,\overline{3};\ 18\,333,\overline{3}\right).$$

☞ ***Example 8.*** **Chelm Game.**

The municipal government of Chelm is run by a City Council and a Mayor. The Council consists of six Aldermen and a Chairman. A bill can become a law in Chelm in two ways:

- A majority of the Council (with the Chairman voting only in the case of a tie among the Aldermen) approves it and the Mayor signs it.
- The Council passes it, the Mayor vetoes it, but at least six of the seven members of the Council then vote to override the veto (in this situation, the Chairman always votes).

Define $v(S) = 1$ for the *winning coalition*, $v(S) = 0$ for the *losing coalition* S.
An 6-tuple

$$(a_M, a_C, a_1, \ldots, a_6),$$

where M denotes the Mayor, C the Chairman and $1, 2, \ldots, 6$ the Aldermen, is an **imputation** if and only if

$$a_M, a_C, a_1, \ldots, a_6 \geq 0 \qquad \text{a} \qquad a_M + a_C + a_1 + \cdots + a_6 = 1.$$

It can easily be shown that the **core** of this game is empty.

Since every coalition consisting at least of six members wins, it is

$$a_M + a_1 + \cdots + a_6 \geq 1$$

and the same inequality holds in all cases when we omit any one of the summands. Since all summands are nonnegative and the sum of all eight values is equal to one, all must be equal to zero, which is a contradiction.

Let us try to find **Shapley vectors** for this game.

Start with the Mayor's value. Non-zero terms in the sum (6.7) are those for which $K \setminus \{S\}$ is a losing coalition but K is a winning coalition (if they remove the Mayor, the Aldermen approve the law but they do not override Mayor's veto). In this case there exist four types of winning coalitions:

1. K contains the Mayor, three Aldermen and the Chairman. There are

   $$\binom{6}{3} = 20$$

   of such coalitions. Since $|K| = k = 5$, the contribution of these sets to the total value of H_M is

   $$20 \cdot \frac{(N-k)!(k-1)!}{N!} = 20 \cdot \frac{(8-5)!(5-1)!}{8!} = 20 \cdot \frac{1}{280} = \frac{1}{14} \, .$$

2. K contains the Mayor and four Aldermen. There are 15 such coalitions and the contribution of these sets to the total value of H_M is

   $$15 \cdot \frac{(8-5)!(5-1)!}{8!} = \frac{3}{56} \, .$$

3. K contains the Mayor, four Aldermen and the Chairman. There are 15 such coalitions and the contribution of these sets to the total value of H_M is

   $$15 \cdot \frac{(8-6)!(6-1)!}{8!} = \frac{5}{56} \, .$$

4. K contains the Mayor and five Aldermen. There are 6 such coalitions and the contribution of these sets to the total value of H_M is

   $$6 \cdot \frac{(8-6)!(6-1)!}{8!} = \frac{1}{28} \, .$$

On the whole,

$$H_M = \frac{1}{14} + \frac{3}{56} + \frac{5}{56} + \frac{1}{28} = \frac{1}{4} \, .$$

Further, consider the Chairman C. In this case there exist two types of winning coalitions:

1. K contains the Mayor, three Aldermen and the Chairman (the vote of Aldermen ends with a tie, the Chairman votes and the Mayor signs).

2. K contains the Mayor and five Aldermen (the proposal will be vetoed but with help of Chairman's vote the veto will be overrided).

There are 20 coalitions of the first type and 6 coalitions of the second type. Thus,

$$H_C = 20 \cdot \frac{(8-5)!(5-1)!}{8!} + 6 \cdot \frac{(8-6)!(6-1)!}{8!} = \frac{3}{28} \; .$$

The sum of all H's is 1, the values for particular Aldermen are obviously the same, thus for each $i = 1, 2, \dots, 6$ we have

$$H_i = \frac{1}{6}(1 - \frac{1}{4} - \frac{3}{28}) = \frac{3}{28} \; .$$

On the whole:

$$\boldsymbol{H} = \left(\frac{1}{4}, \frac{3}{28}, \frac{3}{28}, \dots, \frac{3}{28}\right)$$

It is clear that the Mayor has much greater power than the Chairman and than an Alderman. And it turns that the Chairman's power is exactly equal to that of an Alderman.

6.2.3 Measuring the Power in Politics

Lloyd Shapley (*1923), Martin Shubik (*1926)

A Method for Eval. the Distribution of Power in a Committee System, 1954

The model of a voting situation: cooperative characteristic function form game where a coalition that can pass a bill (winning coalition) is assigned the value 1, the coalition that can not pass a bill (loosing coalition) is assigned the value 0.

How the power of particular voters in the voting game can be measured?

There is a group of individuals all willing to vote for some bill. They vote in order. As soon as enough members have voted for it, it is declared passed, and the member who voted last is given credit for having passed it. Let us choose the voting order of members randomly. Then we may compute how often a given individual is pivotal. This latter number serves to give us our index. (Shapley, Shubik, 1954)

In other words, the **Shapley-Shubik index** of voter i is

$$\varphi_i = \frac{\text{the number of voting orders, in which } i \text{ is pivotal}}{n!}$$

The combinatorial formula for "S-S" index:

$$\varphi_i = \sum_{i \text{ swings for } S} \frac{(s-1)!(n-s)!}{n!}, \qquad s = |S|$$

where a *swing voter* for coalition S means that the coalition S is winning, but the coalition $S \setminus \{i\}$ is not winning.

John F. Banzhaf III.

***Weighted Voting doesn't work: a Mathematical Analysis*, 1965**

The appropriate measure of a legislator's power is simply the number of different situations in which he is able to determine the outcome. More explicitly, in a situations in which there are n legislators, each acting independently and each capable of influencing the outcome only by means of his votes, the ratio of the power of legislator X to the power of legislator Y is the same as the ratio of the number of possible voting combinations of the entire legislature in which X can alter the outcome by changing his vote, to the number of combinations in which Y can alter the outcome by changing his vote. (Bahzhaf, 1965)

In other words:

The voter i's power should be proportional to the number of coalitions for which i is a swing voter. It is convenient to divide this number by the total number of coalitions containing voter i.

Unnormalized Banzhaf index:

$$\beta_i' = \frac{\text{number of swings for voter } i}{2^{n-1}}$$

Normalized Banzhaf index:

$$\beta_i = \frac{\beta_i'}{\sum_i \beta_i'}$$

***One Man, 3,312 Votes: A Mathematical Analysis of the Electoral College*, 1968**

6.2.4 Nucleolus

Let v be a game in characteristic function form with N players, $\boldsymbol{a}$ a given imputation, K a given coalition. The number

$$e(K, \boldsymbol{a}) = v(K) - \sum_{i \in K} a_i \tag{6.9}$$

is called an **excess of the coalition K with respect to the imputation $\boldsymbol{a}$**.

Denote by $\boldsymbol{e}(\boldsymbol{a})$ the vector with $2^N - 1$ components that consists of excesses of all coalitions. Order its components downwards and denote by $\boldsymbol{f}(\boldsymbol{a})$ the vector formed in this way.

Thus we assign a vector $\boldsymbol{f}(\boldsymbol{a})$ to every imputation $\boldsymbol{a}$. On the set of vectors

$$\{\boldsymbol{f}(\boldsymbol{a});\ \boldsymbol{a} \text{ is an imputation}\}$$

consider a lexicographical order. We say that the **imputation $\boldsymbol{b}$ is more acceptable than the imputation $\boldsymbol{a}$** if

$$\boldsymbol{f}(\boldsymbol{b}) \leq_{(\text{lex})} \boldsymbol{f}(\boldsymbol{a}), \tag{6.10}$$

where $\leq_{(\text{lex})}$ is an inequality in lexicographical order, i.e. either the first component of the vector $\boldsymbol{b}$ is less than the first component of $\boldsymbol{a}$, or the first components are equal and the second component of $\boldsymbol{b}$ is less than the second component of $\boldsymbol{a}$, or both first and second components are equal and the third component of $\boldsymbol{b}$ is less than the third component of $\boldsymbol{a}$, etc.

Notice that if an imputation $\boldsymbol{b}$ is more acceptable than an imputation $\boldsymbol{a}$, it provokes less objections than the imputation $\boldsymbol{a}$, or these objections are equal – the first different excess must be less in $\boldsymbol{f}(\boldsymbol{b})$ than in $\boldsymbol{f}(\boldsymbol{a})$.

Definition 8. Nucleolus of the game is defined as such an imputation for which

$$\boldsymbol{f}(\boldsymbol{b}) \leq_{(\text{lex})} \boldsymbol{f}(\boldsymbol{a}) \qquad \text{for all imputations } \boldsymbol{a}.$$

☛ ***Example 9.*** For the game with characteristic function

$$v(Q) = 0, \quad v(\emptyset) = 0,$$
$$v(\{1\}) = v(\{2\}) = v(\{3\}) = -1,$$
$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 1, .$$

the vector $\boldsymbol{e}(\boldsymbol{a})$ has the following components:

$$\begin{array}{l} -(a_1 + a_2 + a_3), \\ 1 - a_1 - a_2, \\ 1 - a_1 - a_3, \\ 1 - a_2 - a_3, \\ -(1 + a_1), \\ -(1 + a_2), \\ -(1 + a_3). \end{array}$$

The first component is equal to zero because $v(Q) = a_1 + a_2 + a_3 = 0$. Since $a_i \geq v(\{i\}) = -1$, the last three components are always nonpositive. Only two-player coalitions can have a positive excess. The maximum

$$\max_{\boldsymbol{a} \text{ is an imputation}} \{1 - a_1 - a_2,\ 1 - a_1 - a_3,\ 1 - a_2 - a_3\}$$

is attained for $\boldsymbol{a} = (0, 0, 0)$.

Thus, the nucleolus is the imputation $(0, 0, 0)$.