7 INDIVIDUAL AND GROUP DECISION MAKING

7.1 INDIVIDUAL DECISION UNDER UNCERTAINTY

- **Laplace Principle**

  suggests choosing a strategy which is optimal in a situation where the opponent chooses all strategies with equal probabilities. In other words, according to the Laplace Principle, the best that we can do under uncertainty is to behave as under risk, where all strategies of the opponent might appear with equal probabilities.

  In the case of a matrix game given by the matrix $A = (a_{ij})$, the optimal decision according to the Laplace Principle is to choose row i for which

  $$\frac{a_{i1} + a_{i2} + \cdots + a_{in}}{n}$$

  is maximal.

- **Minimax Principle**

  suggests that under uncertainty the intelligent player should choose a strategy which is optimal in a situation where the opponent applies the worst possible strategy.

  In the above notation, the optimal decision according to the Minimax Principle is to choose row i for which

  $$\min_j a_{ij}$$

  is maximal.

- **Principle of Maximin Regret**

  This principle is based on an observation that in many practical situations the quality of a decision is judged ex post without taking into account that in the time when the decision was made the decision maker had not possessed the information on actions of the opponent. The Principle of Maximin Regret protects the decision maker against these ex post objections. To find a decision optimal according to this principle, we calculate first a **matrix of regrets** by subtracting from each element in A the maximal element in the column in which the element lies. To follow the common intuition that small regret is better than big, we change the signs of the matrix's elements. In each row of this matrix of regrets we find out the maximal regret and as an optimal decision we choose the row in which this maximum is minimal.

  In the above notation, the optimal decision according to the Principle of Maximin Regret is to choose row i for which

  $$\max_j \left[a_{ij} - (\max_k a_{kj})\right]$$

  is minimal.

Example 1. Chemical Products Ltd. considers a contract to produce AIDS testing sets. They may sign a contract for 2 000, 3 000, 4 000 or 5 000 testing sets or not engage in the business at all. The production costs for the series of tests are 20 000 EUR, 25 000 EUR, 30 000 EUR and 35 000 EUR, respectively. Before the sets are sent to hospitals, they must pass through destructive random sampling tests. If these tests find that less than 2% of the sets give false results, the price of one set is 20 EUR. If the percentage of defective results lies between 2% and 4%, the price of one set is 10 EUR. If there are more than 4% defective sets, the price of one set is 2 EUR. Chemical Products Ltd. never produced AIDS testing sets before, so it is not possible to assess the quality of the product before the series is produced and sampling tests are materialised. What is the best decision?

Solution

The situation can be described by the following matrix game where the elements in the matrix represent the net profit of the firm in thousands of EUR.

	Defective		
Series	Less than 2%	2 − 4%	More than 4%
0	0	0	0
2 000	20	0	-16
3 000	35	5	-19
4 000	50	10	-22
5 000	65	15	-25

Using the Laplace Principle, we find the maximum of the row averages for the above matrix, that is

$$\max\{0,\ 4/3,\ 7,\ 38/3,\ 55/3\} = 55/3.$$

The best decision is therefore to produce a series of 5 000 testing sets.

Using the Minimax Principle, we find the maximum of the worst possible row profits

$$\max\{0,\ -16,\ -19,\ -22,\ -25\} = 0,$$

that is, the best decision is not to go into the business at all.

Using the Principle of Maximin Regret, we need the matrix of regrets

$$\begin{pmatrix} 65 & 15 & 0 \\ 45 & 15 & 16 \\ 30 & 10 & 19 \\ 15 & 5 & 22 \\ 0 & 0 & 25 \end{pmatrix}$$

The worst row regrets are

$$65,\ 45,\ 30,\ 22,\ 25.$$

The minimal regret may be expected when we produce a series of 4 000 testing sets.

7.2 GROUP DECISION MAKING

Terminology:

- **Let $\mathcal{A} = \{x, y, \ldots, z\}$ be a set of alternatives**
- Let the **individuals** of the society be denoted by 1,2,. . . , i, . . . ,n
- For each individual i and any alternatives u and v, one and only one of the following holds:
  - i prefers u to v, which is written as uP_iv
  - i prefers v to u, which is written as vP_iu
  - i is indifferent between u and v, which is written as uI_iv

Definition 1. By a **profile of preference orderings** for the individuals of the society we mean an n-tuple of orderings, $(R_1, \ldots, R_n)$, where R_i is the preference ordering for the ith individual.

Definition 2. By a **social welfare function** we mean a rule which associates to each profile of preference orderings a preference ordering for the society itself.

Condition 1.

- *The number of alternatives in $\mathcal{A}$ is greater than or equal to three*
- *The social welfare function F is defined for all possible profiles of individual orderings.*
- *There are at least two individuals*

Condition 2 *(positive association of social and individual values).*

If the welfare function asserts that x is preferred to y for a given profile of individual preferences, it shall assert the same when the profile is modified as follows:

- *The individual paired comparisons between alternatives other than x are not changed*
- *Each individual paired comparison between x and any other alternative either remains unchanged or it is modified in x's favor.*

Condition 3 *(independence of irrelevant alternatives).*

Let $\mathcal{B}$ be any subset of alternatives in $\mathcal{A}$. If a profile of orderings is modified in such a manner that each individual's paired comparisons among the alternatives of $\mathcal{B}$ are left invariant, the social orderings resulting from the original and modified profiles of individual orderings should be identical for the alternatives in $\mathcal{B}$.

Condition 4 *(citizen's sovereignity).*

For each pair of alternatives x and y there is some profile if individual orderings such that society prefers x to y.

Condition 5 *(non-dictatorship).*

There is no individual with the property that whenever he prefers x to y (for any x and y) society does likewise, regardless of the preferences of other individuals.

Theorem 1 (Arrow's Impossibility Theorem).

The conditions 1, 2, 3, 4 and 5 are **inconsistent**.

It means that there does not exist any welfare function which possesses the properties demanded by these conditions.

In other words, if a welfare function satisfies conditions 1, 2 and 3, then it is either imposed or dictatorial.

Example 2. **Condorcet winner** (violates condition 1)

Each individual orders all alternatives according to his preferences ⟶ **„Condorcet winner“:** such alternative x that for any other alternative y the number of voters preferring x to y is greater than the number of voters preferring y to x.

For example: the majority prefers C to B, C to A, B to A.

In this case A is defeated by B and C; the majority prefers C to B, the winner is C.

Another example: Consider three voters with the following preferences:

		Voter		
		X	Y	Z
Preferences Ranking	1.	A	C	B
	2.	B	A	C
	3.	C	B	A

Cycle: $A \succ B$, $B \succ C$, $C \succ A$

Example 3. **Borda winner** (violates condition 3)

Each alternative is assigned the number of points for each voter according to the position in his ranking order: one point if it is the last one, two points if it is the one before the last one, etc., n points if it is the first one, provided n denotes the number of alternatives. The winner is the alternative with the highest number of points.

	V	W	X	Y	Z
1.	**A**	**A**	B	B	C
2.	B	C	C	C	B
3.	C	B	**D**	**D**	**D**
4.	**D**	**D**	**A**	**A**	**A**

Number of points: A ... 11
D ... 8

Group preference: $\mathbf{A \succ D}$

Restriction of the set of alternatives to {A, D} :

	V	W	X	Y	Z
1.	**A**	**A**	**D**	**D**	**D**
2.	**D**	**D**	**A**	**A**	**A**

Number of points: A ... 7
D ... 8

Group preference: $\mathbf{D \succ A}$

Proof of Arrow's Impossibility Theorem

1. Suppose that $V \neq \emptyset$ is a **minimal decisive set**, i.e. there exist alternatives $x, y \in A$, such that V is decisive for (x, y), but no proper subset $V' \subset V$ is decisive for any ordered pair of alternatives.

V exists:

- Q is decisive for any pair of alternatives (so-called **Pareto optimality**; follows from conditions 1–4).
- Individuals can be removed one at a time until the remaining set is no longer decisive for any pair. Then, if $V = \emptyset$, then a pair (x, y) would exist, such that $\emptyset$ would be a decisive set; but in this case $Q = Q \setminus \emptyset$ would not be decisive for (x, y), which is a contradiction.

2. Choose an arbitrary $j \in V$; denote $W = V \setminus \{j\}$, $U = Q \setminus V$ (since $|Q| \geq 2$, at least one of the sets U, W is non-empty). Choose an arbitrary $z \in \mathcal{A}$, $z \neq x, y$. Consider the following profile:

$\{j\}$	W	U
x	z	y
y	x	z
z	y	x

- For all $i \in V = W \cup \{j\}$ it is $x \succ_i y$, hence $x \succ y$.
- It must be also $y \succeq z$ (otherwise W would be decisive for (z, y), which is a contradiction with the minimality of V).
- From transitivity we have: $x \succ z$.
- But j is the only individual, who prefers x to z; since V is minimal, $\{j\}$ can not be a proper subset of V, hence $V = \{j\}$.

3. By now, we have shown that for every $z \neq x$, $\{j\}$ is decisive for (x, z). Now consider any $w \in \mathcal{A}$, $w \neq x, z$. We will show that $\{j\}$ is also decisive for (w, z) and (w, x). Consider the following profiles:

$\{j\}$	U	
w	z	From Pareto optimality, we have: $w \succ x$;
x	w	$\{j\}$ is decisive for (x, z), hence $x \succ z$;
z	x	from transitivity: $w \succ z$, tj.
		$\{j\}$ **is decisive for** (w, z).

$\{j\}$	U	
w	z	$\{j\}$ is decisive for (w, z), thus $w \succ z$;
z	x	from Pareto optimality: $z \succ x$;
x	w	from transitivity: $w \succ x$, i.e.
		$\{j\}$ **is decisive for** (w, x).

We have therefore shown that $\{j\}$ **is decisive for any pair of alternatives – thus it is a dictator from the condition 5.**

Remark. Simple majority principle is the only one satisfying the following conditions:

- **Decisiveness:** For any profile of individual choices, it specifies a unique group decision for each paired comparison.
- **Anonymity:** It does not depend upon the labeling of individuals.
- **Neutrality:** It does not depend upon the labeling of the two alternatives.
- **Positive responsiveness:** If for a given profile the rule specifies that $x \succeq y$ and if a single individual then changes his paired comparison in favor of x, while the remainder of the society maintain their former choices, then the rule requires that in the group decision it is $x \succ y$.

Denote $N_x = |\{i \in Q; x \succeq_i y\}|$, $N_y = |\{i \in Q; y \succeq_i x\}|$,
$N_I = |\{i \in Q; x \approx_i y\}|$.

- **Anonymity:** group decision upon x, y depends only upon N_x, N_y, N_I,
- from **neutrality** it follows: $x \approx y$, iff $N_x = N_y$,
- by a repeated use of a **positive responsiveness** it is possible to show:
  $x \succ y$ if and only if $N_x > N_y$, resp.
  $y \succ x$ if and only if $N_y > N_x$.