1 INTRODUCTION: EXPLICIT AND NORMAL FORM GAMES

1.1 EXPLICIT FORM GAMES

Let us illustrate the basic concepts by an example.

☛ ***Example 1.*** **Game of Nim**

Consider a simple game where two players – let us denote them 1, 2 – have two piles at the table in front of them, each consisting of two beans. Player 1 has to take one or two beans away from one pile (the beans can not be returned back). Then there is a second player's turn: he has to take one or two beans from one pile, too. In this way the players take turns, until one of them takes away the last bean – and this player loses.

Provided you could choose whether you have the first or the second turn, what would you decide for?

The game can be represented by the model named **explicit form game** or **game tree**.

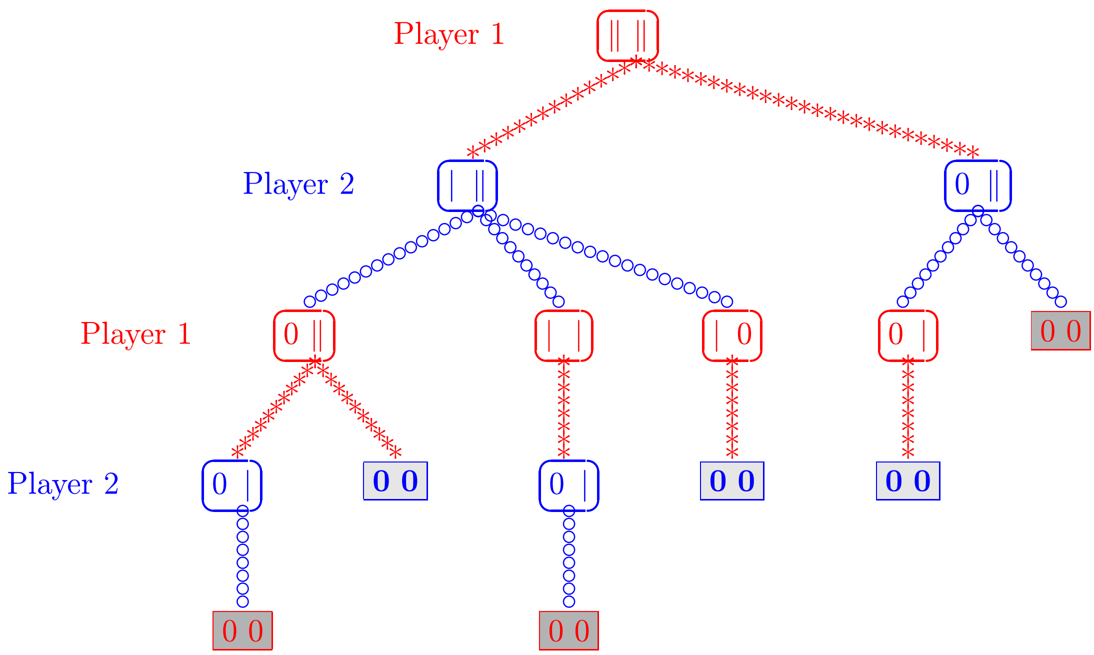


FIG. 1.1: GAME OF NIM

This model shows all situations that can occur in the game. To each situation one **node** corresponds, from each node a certain number of **edges** comes out, that correspond to possible decisions, so called **turns** of a given player. If a player decides for some turn, then he induces a new situation in which the second player decides – to this new situation another node corresponds, that is connected with the previous one by an edge.

Drawing the tree, we usually proceed from the top downwards or from left to right, alternating the first player's and second player's nodes regularly.

There is always just one node with the property that no edge enters it; such node is called an **initial node** or **root** of the tree. Further, there are nodes from which no edges go out; these nodes are called **terminal nodes** and they correspond to positions in which the result of the game is clear and the game ends.

From Figure 1.1 it is obvious that whatever strategy the first player chooses, the second player can choose a strategy that leads him to the victory.

☛ ***Example 2.*** **Game of Nim – Modification**

In the game from Example 1, consider three piles instead of two, each consisting of two beans again; the rules of the game are the same. Which player has a winning strategy?

Hint: The first player can take one pile away from the table and hence force the opponent to the position of the first player in the previous variant with two piles.

☛ ***Example 3.*** **Voting on Wage Rise**

Three legislators vote of their own wage rise. Each of them wishes the rise. Nevertheless, together with voting "YES" a legislator faces up to a loss of voter's favour worth c. The benefit b from the rise exceeds the loss c, $b > c$.
Provided the legislators vote successively and publicly, is it better to be first or last in the election? The last one can see what the situation is like and can possibly decide whether the rise passes or not. Is it therefore the most advantageous?
The situation can be depicted by the following figure:

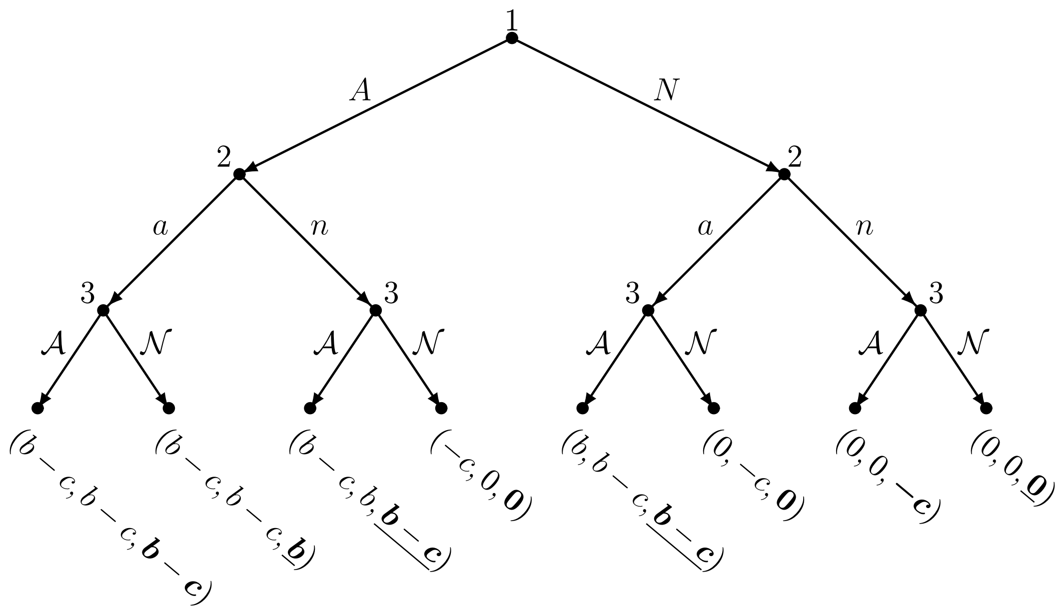
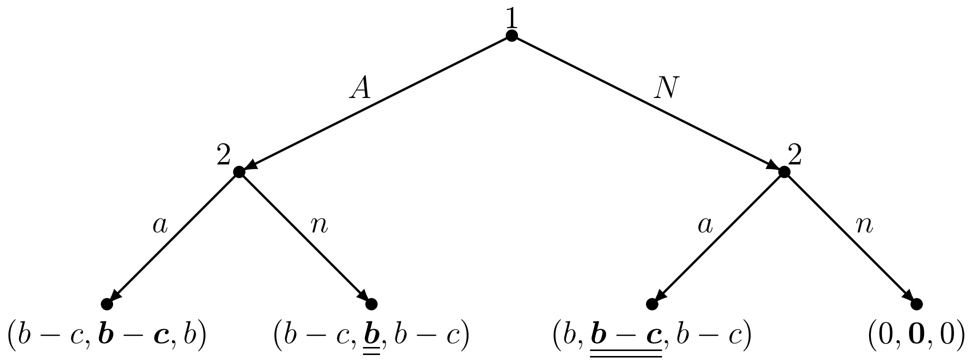


Fig. 1.2: Voting on Wage Rise

The number by a node expresses which legislator's turn it is. The triplet of numbers by each of the terminal nodes expresses the profit of the first, second and third legislator respectively.

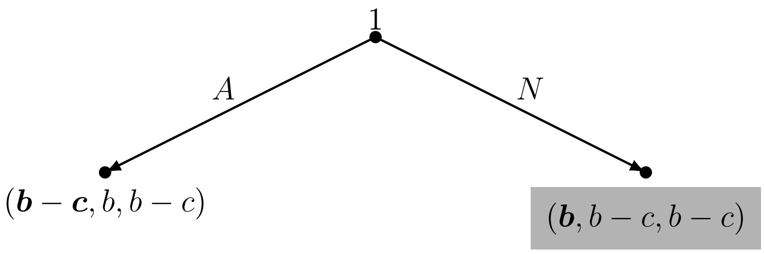
Proceed in the tree from the bottom upwards. In the nodes with number 3 the third legislator decides, whose profit is given by the third number in the triplet. If a situation corresponding to the very left node with number 3 occurs, the third legislator decides between the third numbers in triplets $(b-c, b-c, \boldsymbol{b-c})$ and $(b-c, b-c, \underline{\boldsymbol{b}})$; since $\boldsymbol{b} > \boldsymbol{b-c}$, it is clear that he chooses $(b-c, b-c, \underline{\boldsymbol{b}})$. In the same way we can go through all the nodes with number 3 and label an outcome chosen by the third legislator (in the figure underlined).

The second legislator therefore chooses from the following alternatives in each of his nodes:



Profits of the second legislator are expressed by second numbers in triplets, more convenient alternatives are underlined by double lines.

The first legislator can consider the choices of his colleagues in particular situations in advance and he can see that, strictly speaking, he decides between two possibilities:



More advantageous is obviously the alternative on the right. Hence, if the first legislator votes "NO", the wages will rise anyway and the loss resulting from voting "YES" is carried by the others.

The described reasoning is called **backward induction** – on the base of anticipating the future, the most convenient alternatives are deduced at the beginning of the decision.

☛ ***Example 4.*** **Two-Stages Committee Voting**

Martin, Peter and Paul are the membership committee of the very exclusive Sharebroker Society. The final item on their agenda one morning is a proposal that Alice should be admitted as a new member. No mention is made of another possible candidate called David, and so an amendment to the final item is proposed. The amendment states that Alice's name should be replaced by David's. The rules for voting in committees call for amendments to be voted on in the order in which they are proposed. The committee therefore begins by voting on whether David should replace Alice. If Alice wins, they then vote on whether Alice or Nobody should be made a new member. If David wins, they then vote on whether David of Nobody should be made a new member.
Preferences of particular members are the following:

Ranking	**Martin**	**Peter**	**Paul**
1.	Alice	Nobody	David
2.	Nobody	Alice	Alice
3.	David	David	Nobody

If everybody just voted according to their rankings, the election would go off in the following way: in a vote between Alice and David, Alice would win because both Martin and Peter rank Alice above David and so Paul would be outvoted. Thus, if there is no strategic voting, Alice will be elected to the club because she will also win when she is matched against Nobody.

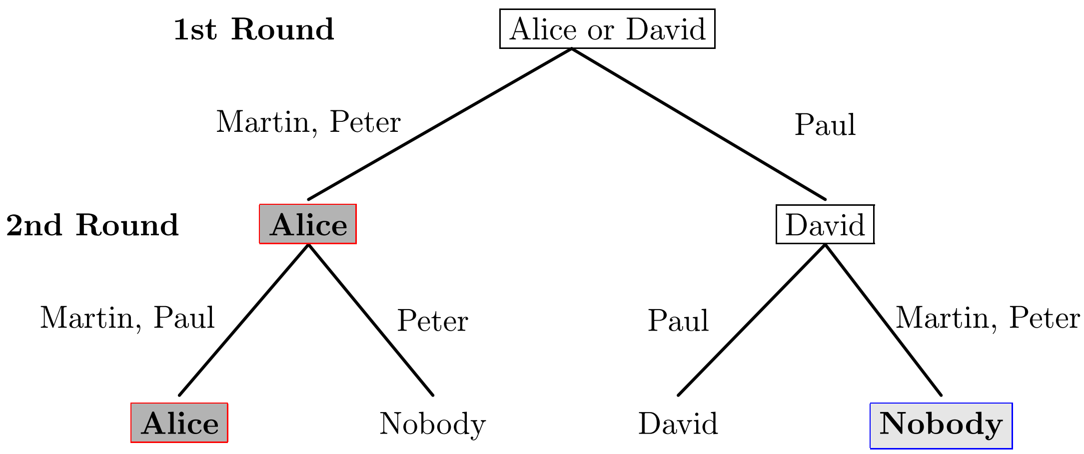


Fig. 1.3: Two-Stages Committee Voting

However, if Peter looks ahead, he will see that there is no point in voting against David at the first vote. If David wins the first vote, then Nobody will triumph at the second vote, and Nobody is Peter's first preference. Thus, Peter should switch his vote from Alice at the first vote, and cast his vote instead for David, who is the candidate he likes the least. If Paul and Martin do not also vote strategically, the result will be that Nobody is elected.

But Paul may anticipate that Peter will vote strategically and he can vote strategically, too, by switching his vote from David to Alice; he thereby ensures that Alice is elected rather than Nobody that is his least desired alternative.

☛ ***Example 5.* Sophisticated Voting in Various Juridical Systems**
Consider three juridical systems in which three judges decide:

1. **Status Quo** (used e.g. in the USA): First the guilt or innocence of the defendant is decided, then, in the case of the guilt, the punishment is decided.

2. **Roman Tradition**: After hearing the evidences, the judges vote downwards from the most severe sentence to the mildest one (possibly the release). For example, first they vote on whether to impose a death sentence or not; if not, whether to impose a life prison or not, etc.

3. **Mandatory System**: First the sentence for the given crime is stipulated and then it is decided whether the defendant is found guilty.

For the case of simplicity, consider three possible outcomes, death sentence, life sentence and release, and the following preferences of particular judges:

Ranking	**Judge *A***	**Judge *B***	**Judge *C***
1.	Death Sentence	Life Sentence	Release
2.	Life Sentence	Release	Death Sentence
3.	Release	Death Sentence	Life Sentence

1. Status quo

In the first round the judges vote on defendant's guilt or innocence; if everybody just voted according to their rankings, "guilty" would win (judges A, B); in the second round, in the vote between life sentence and death sentence, death sentence would win (judges A, C). The first round is therefore in fact a vote between release and death sentence. Hence in the sophisticated voting, "release" therefore wins the first round (besides judge C, judge B will also vote for "release" in the first round since otherwise his less preferred outcome would occur).

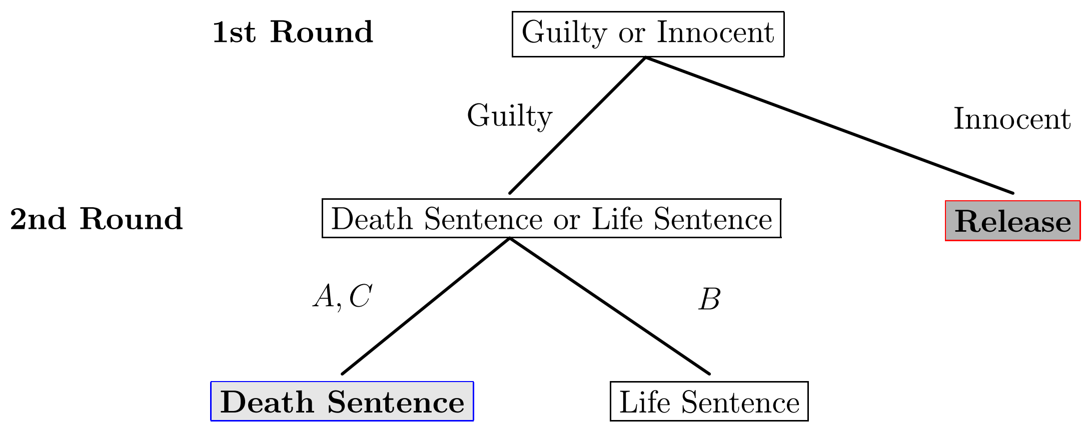


Fig. 1.4: Sophisticated Voting in Status Quo System

2. Roman Tradition

The first round is the vote on the most severe sentence, i.e. whether to impose the death sentence or not. If yes, the sentence is executed, if not, the second round occurs where the judges vote on life sentence or release.

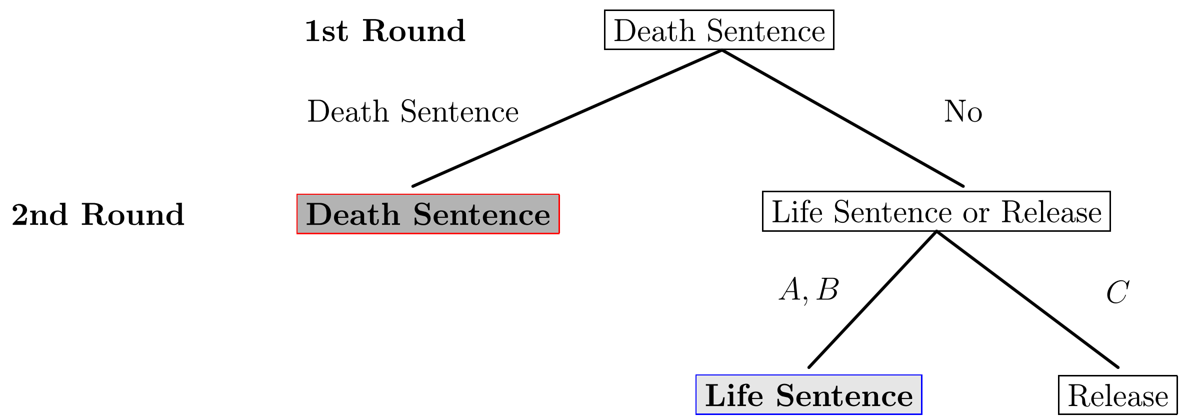


Fig. 1.5: Sophisticated Voting in Roman Tradition

Since in the second round life sentence would win (judges A, B), the first round is in fact a vote between death sentence and life sentence – in the sophisticated voting therefore **death sentence** wins (besides the judge A the judge C will vote for death sentence in the first round, since otherwise the second round would yield his less preferred outcome).

3. Mandatory System

The first round is a vote on the sentence for the given crime, in this case whether to impose death sentence or life sentence. The second round is a vote on whether to impose that sentence or not (release). In the decision between death sentence and release the second one would win (B, C), in the decision between life sentence and release life sentence would win (A, B). The first round is therefore a vote between release and life sentence, hence the defendant will be imposed the life sentence (A will vote for life sentence in the first round to avoid the less preferred outcome: release in the second round).

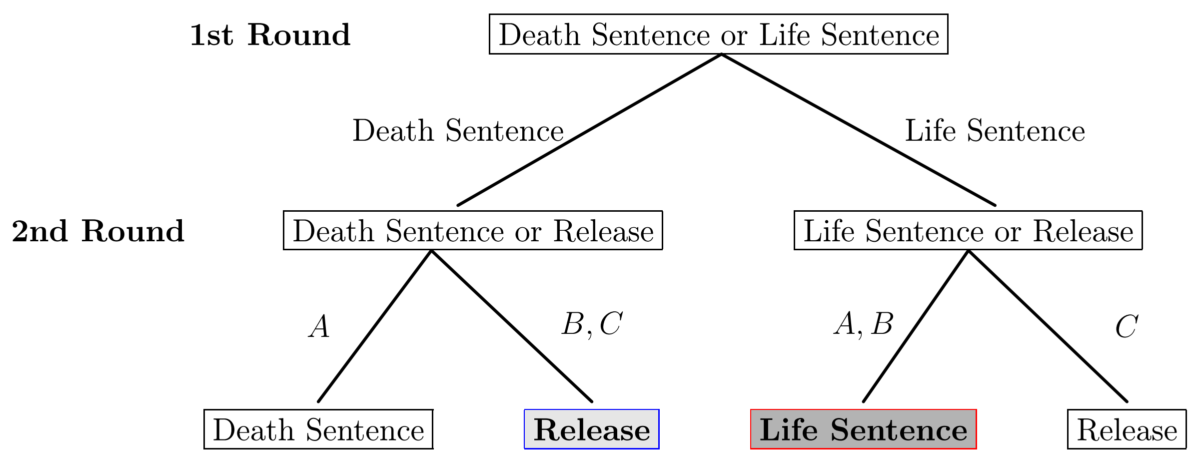


Fig. 1.6: Sophisticated Voting in Mandatory System

1.2 NORMAL FORM GAME

1.2.1 Fundamental Concepts

Definition 1. ***n*-player normal form game** is defined as the $(2n+1)$-tuple

$$(Q;\; S_1, S_2, \ldots, S_n;\; u_1(s_1, s_2, \ldots, s_n), u_2(s_1, s_2, \ldots, s_n), \ldots, u_n(s_1, s_2, \ldots, s_n))\,, \quad (1.1)$$

where $n \geq 2$ is a natural number; $Q = \{1, 2, \ldots, n\}$ is a given finite set, so-called **set of players**, its elements are called **players**; for every $i \in \{1, 2, \ldots, n\}$, S_i is an arbitrary set, so-called **set of strategies of the player** i, and

$$u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$$

is a real function called **payoff function of the player** i.

Definition 2. An n-tuple of strategies $\boldsymbol{s}^* = (s_1^*, \ldots, s_n^*)$ is called an **equilibrium point** or **Nash equilibrium** of the game (1.1), if and only if for every $i \in \{1, 2, \ldots, n\}$ end every $s_i \in S_i$ the following condition holds:

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \leq u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*). \quad (1.2)$$

1.2.2 Finite Normal Form Game

Finite normal form game is a game (1.1) where all the sets $S_1, S_2, \ldots, S_n$ are finite.

Definition 3. Consider a finite n-player normal form game. Denote the number of elements of the strategy set S_i of an arbitrary player i by the symbol m_i. **Mixed strategy** of the player i is defined as the vector of probabilities

$$\boldsymbol{p}^i = (p_1^i, p_2^i, \ldots, p_{m_i}^i), \qquad \text{with} \quad p_j^i \geq 0 \quad \text{for all} \quad 1 \leq j \leq m_i,$$
$$\sum_{j=1}^{m_i} p_j^i = 1, \quad (1.3)$$

where p_j^i, $j = 1, 2, \ldots, m_i$, expresses the probability of choosing the j-th strategy from the strategy space S_i.

A mixed strategy is therefore again a certain strategy that can be characterized in the following way:

"Use the strategy $s_1^i \in S_i$ with the probability p_1^i,
. . . ,
use the strategy $s_{m_i}^i \in S_i$ with the probability $p_{m_i}^i$."

For the case of distinction, the elements of the strategy set S_i are called **pure strategies**.

Theorem 1 (J. Nash). *In mixed strategies, every finite normal form game has at least one equilibrium point.*

1.2.3 Normal Form Games whose Strategy Sets are Open Intervals

Theorem 2 – EQUILIBRIUM TEST.
Let G be a normal form game whose strategy sets are open intervals and payoff functions are twice differentiable. Assume that a strategy profile $(s_1^*, \ldots, s_n^*)$ satisfies the following conditions:

$$\text{1)} \qquad \frac{\partial u_i(s_1^*, \ldots, s_n^*)}{\partial s_i} = 0 \quad \text{for each } \; i = 1, 2, \ldots, n.$$

2) Each s_i^* is the only stationary point of the function

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*), \quad s_i \in S_i.$$

$$\text{3)} \qquad \frac{\partial^2 u_i(s_1^*, \ldots, s_n^*)}{\partial s_i^2} < 0 \quad \text{for each } \; i = 1, 2, \ldots, n.$$

Then $(s_1^*, \ldots, s_n^*)$ is an equilibrium point of the game G.

Remark. In practice, we usually find the solution of the system of equations

$$\frac{\partial u_i(s_1^*, \ldots, s_n^*)}{\partial s_i} = 0 \qquad \text{for each } \; i = 1, 2, \ldots, n,$$

and then use other economic considerations to verify that the solution is the Nash equilibrium of the game.

A typical example of the normal form game whose strategy sets are open intervals is an oligopoly model where several firms produce the same product, each of them contributing significantly to the total production. The price is given by the demand equation which describes the market behavior and gives the relation between the price and the total number of products that should be sold. In other words, it gives the highest price for which it is possible to sell the given amount of products. In Cournot models discussed below the demand equation has the simplest possible form:

$$p + q = M, \qquad M >> c, \tag{1.4}$$

where p is the price of the product, q is the demand for this product in the market, c expresses the production costs of one piece, M is a constant much greater than c.

☛ ***Example 6*** – **Cournot Monopoly Model**

If only one firm – a monopolist – produces the given product, the situation is clear. The monopolist knows that when he produces q products, the highest price for which he can sell one piece to sell all the production out is given by the demand equation (1.4):

$$p = M - q. \tag{1.5}$$

Since nobody else influences the total amount of products, the monopolist is faced with the problem of a mere profit maximization, i.e. finding the maximum of one variable function

$$u(q) = p \cdot q - c \cdot q = Mq - q^2 - cq = (M - q - c)q. \tag{1.6}$$

Maximum can easily be found with the help of a derivative:

$$\begin{aligned} u'(q) = M - c - 2q = &\ 0 \\ q^*_{mon} = &\ \underline{\underline{\tfrac{1}{2}(M - c)}} \end{aligned} \tag{1.7}$$

Since for $q < q^*_{mon}$ the derivative is positive and the function u increasing, for $q > q^*_{mon}$ is the derivative negative and the function u decreasing, it is really a maximum. Hence the monopolist reaches the maximal profit by the production of $q^*_{mon} = \frac{1}{2}(M - c)$ pieces, namely

$$u^*_{mon} = u(q^*_{mon}) = \left(M - \tfrac{1}{2}(M - c) - c\right) \tfrac{1}{2}(M - c) = \underline{\underline{\left[\tfrac{1}{2}(M - c)\right]^2}} \tag{1.8}$$

The corresponding price is then

$$p^*_{mon} = \underline{\underline{\tfrac{1}{2}(M + c)}} \tag{1.9}$$

☛ ***Example 7*** – **Cournot Duopoly Model (1938).**
Now consider two firms producing the same product, each of them contributing significantly to the total production. Monopolist's problem lied in finding the maximum of a simple quadratic function Duopolists are faced with a game, since each of them controls only a part of the total production; the price he receives for his products therefore depends not only on his own decision, but on the decision of the opponent, too. Now we assume that the duopolists decide simultaneously and independently of each other.

Let us find an equilibrium point in this game, i.e. an optimal amounts that should be produced by particular duopolists, when it is not advantageous for none of them to deviate.

Denote q_1, q_2 the production of the first and the second duopolist respectively. Maximal price for which all the products can be sold is again given by the demand equation (1.4):

$$p = M - q_1 - q_2 \tag{1.10}$$

The situation can be modelled by the normal form game where **players** are duopolists, each of them choosing in general the number of an interval $(0, M)$; **strategy sets** are therefore the intervals

$$S_1 = S_2 = (0, M),$$

payoff functions are the profits:

$$\begin{aligned} u_1(q_1, q_2) &= (p - c)q_1 = (M - c - q_1 - q_2)q_1 \\ u_2(q_1, q_2) &= (p - c)q_2 = (M - c - q_1 - q_2)q_2 \end{aligned} \tag{1.11}$$

Although the strategy sets are infinite, the equilibrium point can be found very simply. The first duopolist looks for a function which to each opponent's strategy, i.e. each number q_2 assigns such a number $q_1 = R_1(q_2)$, which is the best reply to q_2 in the sence that the value of the function $u_1(q_1, q_2)$ is maximal. In other words, for every fixed (but arbitrary) $q_2 \in S_2$ the first duopolist searches the maximum of the function $u_1(q_1, q_2)$ which is now a function of one variable q_1 :

$$\frac{\partial u_1}{\partial q_1} = M - c - q_2 - 2q_1 = 0$$

$$R_1(q_2) = q_1 = \tfrac{1}{2}(M - c - q_2) \tag{1.12}$$

(verify this is really a maximum). Similarly, for every strategy q_1 of the first duopolist, the second duopolist searches his best reply $q_2 = R_2(q_1)$, i.e. such number which for a given q_1 maximizes the profit u_2 :

$$\frac{\partial u_2}{\partial q_2} = M - c - q_1 - 2q_2 = 0$$

$$R_2(q_1) = q_2 = \tfrac{1}{2}(M - c - q_1) \tag{1.13}$$

Functions $R_1(q_2)$ and $R_2(q_1)$ are called **reaction curves**; we can represent them in the following way:

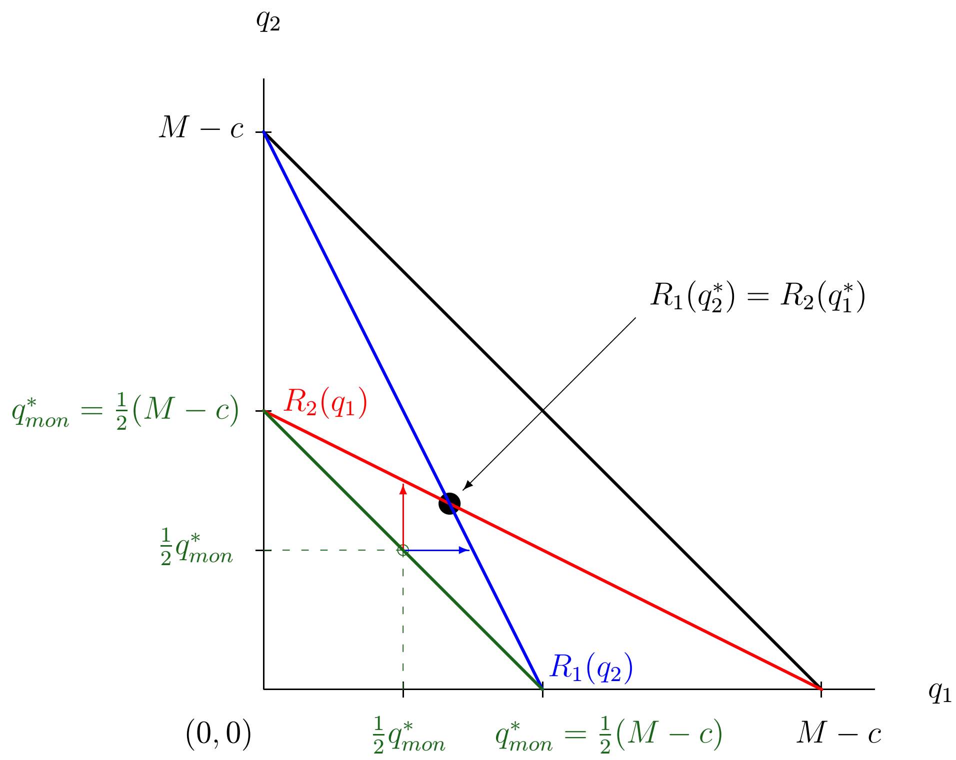


Fig. 1.7: Reaction Curves for Cournot Duopoly

From the definition it can be easily deduced that (q_1^*, q_2^*) is an equilibrium point if and only if it is an intersection of reaction curves:

$$q_1^* = R_1(q_2^*) \qquad \text{end} \qquad q_2^* = R_2(q_1^*).$$

In our example:

$$(q_1^*, q_2^*) = \underline{\underline{\left(\tfrac{1}{3}(M-c), \tfrac{1}{3}(M-c)\right)}}. \tag{1.14}$$

The price for which duopolists will sell the products is then

$$p = M - \tfrac{2}{3}(M-c) = \underline{\underline{\tfrac{1}{3}M + \tfrac{2}{3}c}}. \tag{1.15}$$

The corresponding profit for each duopolist is

$$u_1(q_1^*, q_2^*) = u_2(q_1^*, q_2^*) = \underline{\underline{\left[\tfrac{1}{3}(M-c)\right]^2}}. \tag{1.16}$$

In equilibrium the total production of duopolists is

$$q_D^* = q_1^* + q_2^* = \tfrac{2}{3}(M-c) > \tfrac{1}{2}(M-c) = q^*_{mon}\ ,$$

hence they sell the greater total production for the lower price than a monopolist.

Comparing the results for the monopoly and duopoly, it is clear that it would be the best for duopolists to get together in a cosy cartel and collude to avoid the inroads into their profits that competition brings. Then they could together produce only

$$q_1 + q_2 = q^*_{mon} = \tfrac{1}{2}(M-c) \tag{1.17}$$

(these points form the green line segment) and with respect to circumstances divide the profit that emerged in this way – in symmetrical situation fifty-fifty:

$$\left(\tfrac{1}{2}q^*_{mon}, \tfrac{1}{2}q^*_{mon}\right) = \left(\tfrac{1}{4}(M-c), \tfrac{1}{4}(M-c)\right).$$

Nevertheless, this outcome is **unstable**, because for each duopolist it is profitable to deviate to his best reply to the opponent's choice and receive more for himself.

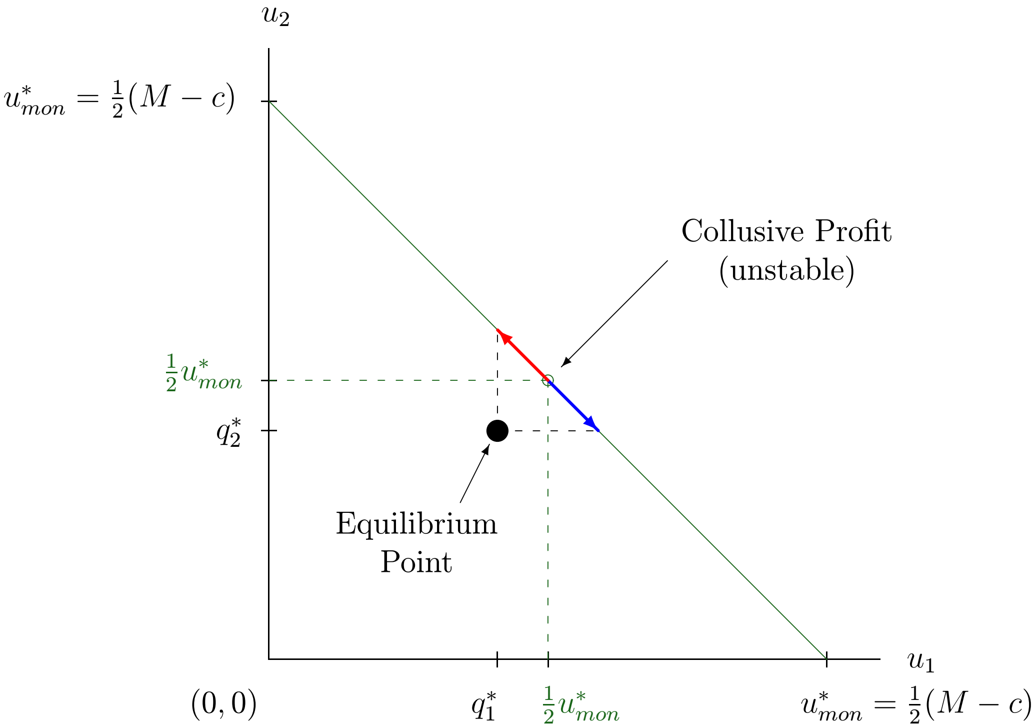


Fig. 1.8: Profits in Cournot Duopoly

The problem lies in the fact that such deals are collusive and, with respect to anti-monopoly laws, usually lawless – and a collusive deal made in smoke-filled hotel room is cheap and certainly not binding.

In the end – fortunately for the customer (and for this reason there are those anti-monopoly laws) – the only deal when no duopolist has a temptation to deviate, is the above mentioned **equilibrium point**

$$(q^*_1, q^*_2) = \left(\tfrac{2}{3}q^*_{mon}, \tfrac{2}{3}q^*_{mon}\right) = \left(\tfrac{1}{3}(M-c), \tfrac{1}{3}(M-c)\right).$$

But the situation changes radically when the game is **repeated**, i.e. when the same duopolists find themselves in the same situation repeatedly: if the probability that there will be one more round is high in each time the game is played, it can be advantageous for each duopolist to keep the deal.

☛ ***Example 8*** – **Cournot Oligopoly Model**
Consider n firms producing the same product, each of them contributing significantly to the total production. Now we have an n-player game where each player searches the optimal amount of products q_i he shall produce. Let us find the equilibrium point of this game.

Likewise in the case of duopoly, profits of particular oligopolists are the following:

$$\begin{aligned} u_1(q_1, q_2, \ldots, q_n) &= (p-c)\, q_1 = (M-c-q_1-q_2-\cdots-q_n)\, q_1 \\ u_2(q_1, q_2, \ldots, q_n) &= (p-c)\, q_2 = (M-c-q_1-q_2-\cdots-q_n)\, q_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_n(q_1, q_2, \ldots, q_n) &= (p-c)\, q_n = (M-c-q_1-q_2-\cdots-q_n)\, q_n \end{aligned} \tag{1.18}$$

Again, it is easy to find the equilibrium point. From the conditions

$$\begin{aligned} \frac{\partial u_1}{\partial q_1} &= M-c-2q_1-q_2-\cdots-q_n = 0 \\ \frac{\partial u_2}{\partial q_2} &= M-c-q_1-2q_2-\cdots-q_n = 0 \\ &\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial u_n}{\partial q_n} &= M-c-q_1-q_2-\cdots-2q_n = 0 \end{aligned}$$

we get the system of equations:

$$\begin{array}{ccccccccc} 2q_1 & + & q_2 & + & \cdots & + & q_n & = & M-c \\ q_1 & + & 2q_2 & + & \cdots & + & q_n & = & M-c \\ \multicolumn{9}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ q_1 & + & q_2 & + & \cdots & + & 2q_n & = & M-c \end{array}$$

Its solution is:

$$q_1^* = q_2^* = \cdots = q_n^* = \underline{\underline{\frac{M-c}{n+1}}} \tag{1.19}$$

Oligopolists therefore produce the total production

$$q_O^* = q_1^* + q_2^* + \cdots + q_n^* = n\frac{M-c}{n+1} = \underline{\underline{\frac{n}{n+1}(M-c)}} \tag{1.20}$$

From the result it is clear that the greater the number of firms is, the greater is the total production q_O^* and the lower is the price p^* and the total profit u^* of the firms:

$$p^* = \underline{\underline{\frac{1}{n+1}M + \frac{n}{n+1}c}} \tag{1.21}$$

$$u^* = \underline{\underline{\frac{n}{(n+1)^2}(M-c)^2}} \tag{1.22}$$

A limit case of oligopoly where $n \to \infty$ is a **perfect competition**: here the great number of small firms contribute to the total production and no firm can significantly influence the total outcome. The total production is now given by

$$\lim_{n\to\infty} \frac{n}{n+1}(M-c) = \underline{\underline{(M-c)}}, \tag{1.23}$$

the price of a product in the market is equal to production costs c,

$$p^* = M - (M-c) = \underline{\underline{c}}, \tag{1.24}$$

and the profit of particular firms is equal to zero,

$$u^* = \underline{\underline{0}}. \tag{1.25}$$

The following table summarizes the above results:

	Total Production q^*	**Price p^***	**Total Profit u^***
Monopoly	$\frac{1}{2}(M-c)$	$\frac{1}{2}M + \frac{1}{2}c$	$\frac{1}{4}(M-c)^2$
Duopoly	$\frac{2}{3}(M-c)$	$\frac{1}{3}M + \frac{2}{3}c$	$\frac{2}{9}(M-c)^2$
Oligopoly	$\frac{n}{n+1}(M-c)$	$\frac{1}{n+1}M + \frac{n}{n+1}c$	$\frac{n}{(n+1)^2}(M-c)^2$
Perfect Competition	$(M-c)$	c	0