2 MATRIX GAMES

1. We have three weapons systems, A_1, A_2, A_3, the adversary has three sorts of aircrafts, B_1, B_2, B_3. Our aim is to shoot down the aircraft, the opponent's aim is to avert it. When the system A_1 is used, the aircrafts B_1, B_2, B_3 are shoot down with brobabilities $0,9$; $0,4$; $0,2$; for the system A_2, the probabilities are $0,3$; $0,6$; $0,8$; for the system A_3, we have the probabilities $0,5$; $0,7$ and $0,2$. Find optimal strategies for both sides.

2. The army **A** sends two bombers **I** and **II** to the territory of the army **B**. The bomber **I** flies first, **II** in the back. One of them – it is not clear which one in advance – carries a bomb, the second one is only an escort. In the opponent's territory the bombers are attacked by an interceptor of **B**.

   If the interceptor attacks the plain **II**, it is strafed by its guns only, if it attacks the plain **I**, it is strafed by the guns of both. The probability of shooting down the interceptor is 0.3 in the first case and 0.7 in the second case. If the interceptor is not shot down, it destroys the target with the probability 0.6.

   The aim of the bombers is to bomb the target, the aim of the interceptor is to avert it.

   Decide:
   a) for the army **A**: which plain shall carry the bomb;
   b) fir the army **B**: which plain to attack.

3. The side **A** attacks the object, the side **B** defends it.The side **A** has two aircrafts, the opponent **B** has three anti-aircraft cannons. Each plain carries so destructive device that to the object destruction only one plane suffices. The aircrafts of the side **A** can choose from three possible directions to access the object, the opponent can aim any cannon to any of these three direction. The cannon shoots in the chosen direction only, and it does so with the efficiency 1. The side **A** does not know how the cannons are aimed and the opponent does not know where the planes arrive from.

   Find optimal strategies for both sides.

4. Modify the previous example for the case that the side **A** can arrive from four differend directions and the side **B** has four anti-aircraft cannons.

5. Let
$$M = \begin{pmatrix} 3 & 2 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$
   be a matrix of a game. By eliminating the dominated rows and columns reduce the matrix M to the least possible size.

Excercises for the course *Game Theory and Optimal Decisions* at FD ČVUT.
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6. Imagine you are a player whose strategies are given by the columns of the 3×4 matrix and who wants to realize a mixed strategy $\boldsymbol{y} = (5/12, 1/4, 1/3)$. Your calculator generates 8-digit random numbers regularly distributed in $(0, 1)$. Explain how would you use these numbers for realizing the strategy $\vec{p}$ (as precise as possible).

7. Verify that the pair

$$\boldsymbol{x} = (21/52, 3/13, 0, 3/52, 4/13), \quad \boldsymbol{y} = (5/52, 0, 11/52, 17/26, 1/26), \quad \mathbf{v} = 19/52$$

is the solution of the matrix game given by the matrix

$$M = \begin{pmatrix} -1 & 2 & -2 & 0 & 1 \\ -2 & -1 & 3 & 2 & 0 \\ 2 & 1 & 0 & -1 & -2 \\ 0 & 0 & 2 & 1 & 1 \\ 1 & -1 & 0 & -2 & 1 \end{pmatrix}.$$

8. Verify that the pair

$$\mathbf{x} = (3/4, 1/4, 0), \quad \mathbf{y} = (1/2, 0, 1/2), \quad \mathbf{v} = 1/2$$

is the solution of the matrix game given by the matrix

$$M = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}.$$

9. Verify that the pair

$$\mathbf{x} = (0, 1/4, 1/2, 1/4) \qquad \text{a} \qquad \mathbf{y} = (1/2, 0, 1/6, 0, 1/3)$$

is the solution of the matrix game given by the matrix

$$M = \begin{pmatrix} 2 & -3 & 4 & -5 \\ -1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 3 \\ 1 & 2 & -3 & 4 \\ -3 & 4 & -5 & 6 \end{pmatrix}.$$

10. Let M be a 2×2 matrix. Prove that M has a saddle point if and only if it has a dominating row or a dominating column. Is it true for larger matrices, too?

11. Solve the matrix game

$$M = \begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & -1 \\ -2 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}.$$

12. Solve the matrix game

$$M = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix}.$$

13. Solve the matrix game

$$M = \begin{pmatrix} 0 & 1 & 2 & -3 \\ -1 & 0 & 1 & 0 \\ -2 & -1 & 0 & 1 \\ 3 & 0 & -1 & 0 \\ 2 & 0 & -1 & -1 \end{pmatrix}.$$

14. Solve the matrix game

$$M = \begin{pmatrix} 0 & 2 & -1 & -3 \\ -2 & 0 & 3 & -1 \\ 1 & -3 & 0 & 1 \\ 2 & 1 & -1 & 0 \end{pmatrix}.$$

15. Solve the matrix game

$$M = \begin{pmatrix} -3 & 2 & 0 \\ 1 & -2 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & -3 \end{pmatrix}.$$

16. Solve the matrix game

$$M = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

17. Solve the matrix game

$$M = \begin{pmatrix} -1 & 2 & -1 & 1 \\ 1 & 0 & 2 & -1 \\ -1 & 1 & -2 & 2 \end{pmatrix}.$$

18. Solve the matrix game

$$M = \begin{pmatrix} -2 & 3 & 0 & 1 \\ 1 & -3 & 2 & 0 \\ 0 & -2 & 1 & 3 \\ -1 & 1 & -1 & 2 \end{pmatrix}.$$

19. Solve the matrix game

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

20. Solve the game of ”paper, scissors, stone”.