3 UTILITY THEORY

3.1 A CLASSIFICATION OF DECISION MAKING

3.1.1 Decision under Certainty

Definition 1. We say that the decision is taken **under certainty** if each action is known to lead invariably to a specific outcome (prospect, alternative, etc.).

Mathematical tools: the calculus to find maxima and minima of functions, the calculus of variations to find functions, production schedules, inventory schedules, etc.

3.1.2 Decision under Risk

Definition 2. We say that the decision is taken **under risk** if each action leads to one of a set of possible specific outcomes, each outcome occurring with a known probability.

Remark. Certainty is a degenerate case of risk where the probabilities are 0 and 1.

Example 1. An action might lead to a reward of \$10 if a fair coin comes up heads, and a loss of \$5 if it comes up tails.

Example 2. More generally, consider a gamble in which one of n outcomes will occur, and let the possible outcomes be worth $a_1, a_2, \dots a_n$ euros, respectively. Suppose that it is known that the respective probabilities of these outcomes are $p_1, p_2, \dots, p_n$, where each p_i lies between 0 and 1 (inclusive) and their sum is 1. How much is it worth to participate in this gamble?

The monetary expected value: $b = a_1p_1 + a_2p_2 + \cdots + a_np_n$.

Objections to the monetary expected value – St. Petersburg Paradox:

Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. Suppose we seek to determine the value of Paul's expectation.

The mean value of the win in ducats:

$$1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 2^2 \cdot \frac{1}{2^3} + \cdots + 2^n \cdot \frac{1}{2^{n+1}} + \cdots = \infty$$

Paradox: a reasonable person sells – with a great pleasure – the engagement in the play for 20 ducats.

Daniel Bernoulli: a gamble should be evaluated not in terms of the value of its alternative pay-offs but rather in terms of the value of its **utilities**, which he derived to be logarithmic functions.

3.1.3 Decision under Uncertainty

Definition 3. We say that the decision is taken **under uncertainty** if either action has as its consequence a set of possible specific outcomes, but the probabilities of these outcomes are completely unknown or are not even meaningful.

3.2 AXIOMATIC UTILITY THEORY

3.2.1 Rational Preferences

Consider a finite set $\{A_1, A_2, \ldots, A_r\}$ of basic alternatives or prizes. **A lottery**

$$(p_1 A_1, p_2 A_2, \ldots, p_r A_r)$$

is a chance mechanism which yields the prizes $A_1, A_2, \ldots, A_r$ as outcomes with known probabilities $p_1, p_2, \ldots, p_r$, where each $p_i \geq 0$, $p_1 + p_2 \cdots + p_r = 1$. Let us order the alternatives downwards from the most to the least preferred one.

Among the basic alternatives, we use the symbolism $A_i \succsim A_j$ to denote that A_j is not preferred to A_i. Equivalently, we say that A_i is preferred or indifferent to A_j.

Assumption 1 **(ordering of alternatives)**. *The "preference or indifference" ordering over all basic alternatives is complete and transitive: for any A_i and A_j, either $A_i \succsim A_j$ or $A_j \succsim A_i$ holds; and if $A_i \succsim A_j$ and $A_j \succsim A_k$ then $A_i \succsim A_k$.*

Now suppose that $L^{(1)}, L^{(2)}, \ldots, L^{(s)}$ are any s lotteries which each involve $A_1, A_2, \ldots, A_r$ as prizes. If $q_1, q_2, \ldots, q_r$ are any s nonnegative numbers which sum to 1, then

$$\left(q_1 L^{(1)}, q_2 L^{(2)}, \ldots, q_s L^{(s)}\right)$$

denotes a **compound lottery** in the following sense: one and only one of the given s lotteries will be the prize, and the probability that it will be $L^{(i)}$ is q_i.

For the sake of simplification, let us denote A_1 the most preferred alternative, A_r the least preferred one.

Assumption 2 **(reduction of compound lotteries)**. *Any compound lottery is indifferent to a simple lottery with $A_1, A_2, \ldots, A_r$ as prizes, their probabilities being computed according to the ordinary probability calculus. In particular, if*

$$L^{(i)} = \left(p_1^{(i)} A_1, p_2^{(i)} A_2, \ldots, p_r^{(i)} A_r,\right) \qquad \textit{for } i = 1, 2, \ldots, s,$$

then

$$\left(q_1 L^{(1)}, q_2 L^{(2)}, \ldots, q_s L^{(s)}\right) \sim (p_1 A_1, p_2 A_2, \ldots, p_r A_r),$$

where

$$p_i = q_1 p_i^{(1)} + q_2 p_i^{(2)} + \cdots + q_s p_i^{(s)}.$$

Assumption 3 **(continuity)**. *Each prize A_i is indifferent to some lottery involving just A_1 and A_r. That is, there exists a number u_i such that A_i is indifferent to*

$$(u_i A_1, 0A_2, \ldots, 0A_{r-1}, (1-u_i)A_r)\,.$$

For convenience, we write:

$$A_i \sim (u_i A_1,\, (1-u_i)A_r) = \tilde{A}_i.$$

Assumption 4 **(substitutibility)**. *In any lottery L, $\tilde{A}_i$ is substitutable for A_i , that is,*

$$(p_1 A_1, p_2 A_2, \ldots, p_i A_i, \ldots, p_r A_r) \sim (p_1 A_1, p_2 A_2, \ldots, p_i \tilde{A}_i, \ldots, p_r A_r).$$

Assumption 5 **(transitivity)**. *Preference and indifference among lotteries are transitive relations.*

Assumption 6 **(monotonicity)**. *A lottery $(pA_1, (1-p)A_r)$ is preferred or indifferent to $(p'A_1, (1-p')A_r)$ if and only if $p \geq p'$.*

Theorem 1. *If the preference or indifference relation $\succsim$ satisfies assumptions 1 trough 6, there are numbers u_i associated with the basic prizes A_i such that for two lotteries L and L' the magnitudes of the expected values*

$$p_1 u_1 + p_2 u_2 + \cdots + p_r u_r \quad \textit{and} \quad p_1' u_1 + p_2' u_2 + \cdots + p_r' u_r$$

reflect the preference between the lotteries.

Definition 4. If a person imposes a transitive preference relation $\succsim$ over a set of lotteries and if to each lottery L there is assigned a number $u(L)$ such that the magnitudes of the numbers reflect the preferences, i.e., $u(L) \geq u(L')$ if and only if $L \succsim L'$, then we say there exists a **utility function** u over the lotteries.

If, in addition, the utility function has the property that

$$u\left(qL, (1-q)L'\right) = qu(L) + (1-q)u(L')$$

for all probabilities q and lotteries L and L', then we say the utility function is **linear.**