2 HISTORICAL BEGINNINGS OF GAME THEORY

2.1 TALMUD

Rabbi Jehuda ha-Nasi (135 – 217)

Rabbi Jehuda compiled a verbal tradition into a comprehensive collection of post-biblical religious law: *Mishna* (learning, revision). In the following period Mishna was studied, discussed, polemized, new rules were formed, ... On this basis, *Gemara* (addition), an extensive commentary to Mishna, was created. The synthesis of Mishna and Gemara is called ***Talmud***; besides Bible it is one of the fundamentals of Jewish culture and religion.

- *Babylonian* (final revision by Rabbi Ashi, † 427)
- *Jerusalem, Palestine* (completed in the middle of the 4th century, has not remained so well and complete preserved)

Babylonian Talmud: A Bankruptcy Problem

Babylonian Talmud contains a fascinating discussion of a "bankruptcy problem": a man has three wives whose marriage contracts specify that in the case of his death they receive 100, 200, 300 respectively. He dies, leaving the estate $E < 600$. How should the estate be divided among the wives? The Mishna stipulates the following division:

Debts

		100	200	300
	100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
E	200	50	75	75
	300	50	100	150

Aumann, Maschler, 1985: *Game Theoretic Analysis of a Bankruptcy Problem from the Talmud*: the analysis of a general bankruptcy problem with debts $d_1, \ldots, d_n$ and the estate E; they provide three explanations of the table based on Talmudic principles, one explanation based on today cooperative game theory.

1. The Contested Garment (Baba Metzia)
Two hold a garment; one claims it all, the other claims half. Then the one is awarded $\frac{3}{4}$, *the other* $\frac{1}{4}$.

→ 2-creditor bankruptcy problem: **CG principle**

$$x_i = \frac{E - (E - d_1)_+ - (E - d_2)_+}{2} + (E - d_j)_+$$

$$(\alpha)_+ := \mathrm{Max}\,(\alpha, 0)$$

The solution as a function of E ($d_1 \leq d_2$): equal divisions $\rightarrow$ each receives $d_1/2 \rightarrow$ each additional euro to the greater claimant $\rightarrow$ each receives all but $d_1/2$ of her claim $\rightarrow$ each additional euro divided equally

Rabbi Hai Gaon (10. century) Mishna about bankruptcy should be explained on the basis of that in Baba Metzia (no explicit connection)

Bankruptcy problem:

$$(E;d); \quad d=(d_1,\dots,d_n), \quad 0\leq d_1\leq \dots \leq d_n, \quad 0 \leq E \leq d_1+\dots+d_n = D$$

Solution:

$$x=(x_1,\dots,x_n)\in \mathbb{R}^n; \quad x_1+\dots+x_n=E \quad (x_i \text{ – the amount assigned to claimant } i)$$

CG-consistent solution:

$$\forall i \neq j: \quad \text{division of } x_i+x_j \text{ prescribed by the CG principle for claims} d_i, d_j \text{ is } (x_i,x_j).$$

Theorem A. Each bankruptcy problem has a unique consistent solution.
Proof. No more that one: contradiction; at least one:
$\boxed{E\ small}$: equal division $E/n \rightarrow 1$ receives $d_1/2 \rightarrow$ each additional euro equally between $2,\dots,n \rightarrow 2$ receives $d_2/2 \rightarrow$ each additional euro equal between $3,\dots,n, \rightarrow \cdots \rightarrow (d_1/2, d_2/2,\dots,d_n/2) \sim \boxed{E=D/2}$
$\boxed{E\geq D/2}$: mirror process (i's *award* $x_i \rightarrow$ *loss* $d_i - x_i$):
total loss $\boxed{D-E\ small}$: equal division $d_i - \frac{D-E}{n} \rightarrow 1$ loses $d_1/2 \rightarrow$ further losses between $2,\dots,n \rightarrow 2$ loses $d_2/2 \rightarrow \cdots \rightarrow \boxed{E=D/2}$

2. "More/less than half is like the whole/nothing"

Talmudic principle: in general, a lender automatically has a lien on the borrower's property; when his entire property is worth less than half the loan, the borrower may in certain cases dispose of it "free and clear".

Social justice: all creditors at the same side of the watershed given by the half.

3. Coalition Formation and Jerusalem Talmud

Creditors empower each other; the third empowers the second to deal with the first. She may say to her, "Your claim is 100, right? Take 50 and go."
e.g. $200 = 50 + (75+75)$; $300 = 50 + (100+150)$ (+CG)

Consistent solution:

$$\begin{array}{rcccll} 0 & \leq & E & \leq & 3d_1/2 & \Rightarrow \text{ equal division} \\ 3d_1/2 & \leq & E & \leq & D-3d_1/2 & \Rightarrow \text{ coalition 2+3} \\ D-3d_1/2 & \leq & E & \leq & D & \Rightarrow \text{ loss divided equally} \end{array}$$

4. Cooperative Game Theory

$Q = \{1, 2, \ldots, n\}$ – set of players; $S \subseteq Q$ – coalition;
$v: \ S \mapsto v(S) \in \mathbb{R}$ – worth of S; $v(\emptyset) = 0$;
$x = (x_1, \ldots, x_n)$ – payoff vector; x_i – the payoff to i
The game $v_{E;d}$ corresponding to the bankruptcy problem $(E; d)$:

$$v_{E;d} := (E - d(N \setminus S))_+$$

Theorem. The consistent solution of a bankruptcy problem is the nucleolus of the corresponding game.

2.2 THE BIRTH OF PROBABILITY CALCULUS

1654 correspondence:
Blaise Pascal (1623 – 1662) and **Pierre de Fermat (1607 – 1663)**

2.3 LE HER - FIRST MIXED STRATEGY SOLUTION

Peter deals Paul a single card at random from a deck of cards (A, 1, . . . , 10, J, Q, K), then to himself; neither player sees the card dealt to the other

Object: to hold the card of the higher value

Rules:

Paul is not satisfied	$\Rightarrow$	may force Peter to exchange with him (except Peter has a king)
Peter is not satisfied	$\Rightarrow$	may exchange it for a card dealt from the deck at random (except the new card is a king)
Cards of the same value	$\Rightarrow$	Peter wins

Nicholas Bernoulli (1687 – 1759), **Pierre Rémond de Montmort (1678 – 1719)**

Correspondence:

- Paul should change every card < 7, keep > 7
- Peter should change every card < 8, keep > 8

In disputable cases:

N. Bernoulli: both should change
P. de Montmort: no precept can be determined

James Waldegrave (1684 – 1741)

1713 letter to de Montmort: looks for a strategy that **maximizes the probability of player's win, whichever strategy is chosen by the opponent**

- Peter should choose the strategy
  keep 8 and higher with the probability 5/8
  change 8 and lower with the probability 3/8;

- Paul should choose the strategy
  keep 7 and higher with the probability 3/8
  change 7 and lower with the probability 5/8

de Montmort, 1713: Essai d'Analyse sur les Jeux d'Hasard (appendix: correspondence of de Montmort with Jean and Nicholas Bernoulli)

2.4 BEGINNINGS OF UTILITY THEORY

2.4.1 Daniel Bernoulli (1700–1782)

1725–33 Exposition of a New Theory of Risk Evaluation

A gamble should be evaluated not in terms of the value of its alternative pay-offs but rather in terms of the value of its **utilities**.

Illustration example: *Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats? Would he not be ill-advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is negative. On the other hand I am inclined to believe that a rich man would be ill-advised to refuse to buy the lottery ticket for nine thousand ducats. If I am not wrong then it seems clear that all men cannot use the same rule to evaluate the gamble ...*

The attempt to express the utility as a function $u(q)$ giving the number of *utility units* for the financial amount q; assumption: the increase of utility $u(q)$ is proportionate to the increase of the amount dq and inversely proportionate to the quantity previously possessed (a poor man generally obtains more utility than does a rich man from an equal gain):

$$\begin{aligned} du(q) &= \frac{b\,dq}{q} && b > 0 \quad \text{(constant of proportion)} \\ u(q) &= b \ln q + c && c \in \mathbb{R} \\ &= b \ln q - b \ln \alpha && \alpha \in (0, +\infty) \\ u(q) &= b \ln \frac{q}{\alpha} && \alpha - \text{the quantity originally possessed} \end{aligned}$$

Application: elucidation of **St. Petersburg Paradox:**

Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. Suppose we seek to determine the value of Paul's expectation.

The mean value of the win:

$$1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 2^2 \cdot \frac{1}{2^3} + \cdots + 2^n \cdot \frac{1}{2^{n+1}} + \cdots = \infty \tag{2.1}$$

Paradox: although an expected value of the win is infinite, a reasonable person sells – with a great pleasure – the engagement in the play for 20 ducats.

Bernoulli: the mean value of the utility brought by the win:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} b \ln \frac{\alpha + 2^{n-1}}{\alpha} = b \ln[(\alpha+1)^{\frac{1}{2}}(\alpha+2)^{\frac{1}{4}} \cdots (\alpha + 2^{n-1})^{\frac{1}{2^n}} \cdots] - b \ln \alpha \tag{2.2}$$

Amount D, whose addition to the initial possession brings the same utility:

$$b \ln \frac{\alpha + D}{\alpha} = b \ln[(\alpha+1)^{\frac{1}{2}}(\alpha+2)^{\frac{1}{4}} \cdots (\alpha + 2^{n-1})^{\frac{1}{2^n}} \cdots] - b \ln \alpha$$

$$D = [(\alpha+1)^{\frac{1}{2}}(\alpha+2)^{\frac{1}{4}} \cdots (\alpha + 2^{n-1})^{\frac{1}{2^n}} \cdots] - \alpha \tag{2.3}$$

For a zero initial possession:

$$D = \sqrt[2]{1} \cdot \sqrt[4]{2} \cdot \sqrt[8]{4} \cdot \sqrt[16]{8} \cdots = 2$$

Imperfections of Bernoulli's utility function

- Defined for positive values of q only, while in the real world the losses are important, too
- Utility function is different for different people, depends on other than property conditions, too

Important stimulus for further development.

Similar – but independent – considerations (Bernoulli cites at the end of his trearise):

Gabriel Cramer (1704 – 1752)

Letter to Nicholas Bernoulli, 1728

Idea: people evaluate money in proportion to the utility they can obtain from it.

Assumption:

Any amount above 10 millions, or (for the sake of simplicity) above 2^{24} *ducats be deemed by me equal in value to* 2^{24} *ducats or, better yet, that I can never win more than that amount, no matter how long it takes before the coin falls with its cross upward* [without any further discussion]. *In this case, my expectation is*

$$\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 4 + \cdots + \frac{1}{2^{24}} \cdot 2^{24} + \frac{1}{2^{25}} \cdot 2^{24} + \frac{1}{2^{26}} \cdot 2^{24} + \cdots = \\ = \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 12 + 1 = 13\,. \tag{2.4}$$

Thus, my moral expectation is reduced in value to 13 ducats and the equivalent to be paid for it is similarly reduced – a result which seems much more reasonable than does rendering it infinite.

2.5 SEARCHING AN EQUILIBRIUM: COURNOT DUOPOLY

Antoine Augustin Cournot (1801 – 1877)

1838 ***Recherches sur les principes mathématiques de la théorie des richesses***

- Mathematically rigorous exposition of the most of today theory of economical competition, monopoly and oligopoly
- Detailed analysis of monopoly – the concept of a *cost function*, etc.
- Mathematical derivation of the total production that can be chosen by the producer to maximize his profit
- Influence of various forms of taxes and other charges, their influence on the profit of the producer and customers
- Duopoly Model – solution corresponding to ***Nash equilibrium*** introduced more than 100 years later
- Oligopoly Model

2.6 ÉMILE BOREL (1871 – 1956)

1921 ***La théorie du jeu et les équations, intégrales à novau symétrique gauche***
Comptes Rendus 173, 1304–1308

- **The first attempt to mathematize the concept of a game of strategy**
- **Method of Play** ... pure strategy
- **Zero-sum, symmetric, finite 2-player games**

$$(\{1,2\}; S_1 = S = \{s_1, \ldots, s_n\}, S_2 = S; u_1, u_2)$$

$$u_1(s_i, s_j) = -u_2(s_i, s_j) = u_2(s_j, s_i)$$

- **Probability of winning:**

$$\pi_1(s_i, s_j) = \tfrac{1}{2} + \alpha_{ij}\,; \quad \pi_2(s_i, s_j) = \tfrac{1}{2} + \alpha_{ji}$$

$$\alpha_{ij} + \alpha_{ji} = 0\,;\ \alpha_{ii} = 0\,;\ \alpha_{ij}, \alpha_{ji} \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]$$

each player is trying to maximize π_i

bad strategy s_i ... $\exists s_k :\ \forall s_j : \alpha_{ij} \le \alpha_{kj}$
best strategy s_i ... $\forall s_k :\ \forall s_j : \alpha_{ij} \le 0$

Mixed strategies: $\boldsymbol{p} = (p_1, \ldots, p_n),\ \boldsymbol{q} = (q_1, \ldots, q_n)$

$$\pi_1(\boldsymbol{p}, \boldsymbol{q}) = \tfrac{1}{2} + \alpha\,; \quad \alpha = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ji} p_i q_j$$

Solution $\boldsymbol{p}$ ... $\forall \boldsymbol{q} : \alpha = 0$ (minimax solution)

Existence: $n = 3,\ n = 5$ (later) ... proof
$n = 7$ ... hypothesis: yes
$n > 7$ ... hypothesis: no in general

1924 ***Théorie of Probabilités*** (204–224)

α_{ik} = financial amount that II must pay do I

Is it possible for I to choose a mixed strategy such that his payoff is 0, whichever strategy is chosen by II? That is, a mixed strategy which saves him from a negative outcome?

Still he believes: it is NOT possible for $n > 7$, searches a counterexample

1927 ***Sur les systèmes de formes linéaires à déterminant symétrique gauche et la théorie générale du jeu*** Comptes Rendus 184, 52–53

Positive formulation of the problem: *Determine mixed strategies, ...*

No general proof

1938 ***Traité du calcul des probabilités et ses applications***
continuous games (strategy sets: circle etc.)
9 pages: **Jean Ville: the first elementary proof of von Neumann's minimax theorem**

2.7 JOHN VON NEUMANN (1903 – 1957)

1926 proof of **minimax theorem** (Gött.Math.Soc.)

1928 ***Sur la théorie des jeux,*** Comptes Rendus
Zur Theorie der Gesellschaftsspiele, Mat.Ann.

- **Mathematization of strategy games**
- **The proof of "minimax theorem"**

Formulation: **finite n-player zero sum game**

More results: $(\{1, 2\}; \{s_1, \dots, s_k\}, \{t_1, \dots, t_l\}; u_1, u_2)$

Player 2

Player 1

$$\begin{array}{c} \\ s_1 \\ s_2 \\ \vdots \\ s_k \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_2 & \dots & t_l \end{array} \\ \begin{pmatrix} u_1(s_1,t_1) & u_1(s_1,t_2) & \dots & u_1(s_1,t_l) \\ u_1(s_2,t_1) & u_1(s_2,t_2) & \dots & u_1(s_2,t_l) \\ \dots & \dots & \dots & \dots \\ u_1(s_k,t_1) & u_1(s_k,t_2) & \dots & u_1(s_k,t_l) \end{pmatrix} \end{array}$$

Player 1: $\min_{t_j} u_1(s_i, t_j)$ $\rightsquigarrow$ **MAX**
Player 2: $\max_{s_i} u_1(s_i, t_j)$ $\rightsquigarrow$ **MIN**

$$\max_{s_i} \min_{t_j} u_1(s_i, t_j) \leq \min_{t_j} \max_{s_i} u_1(s_i, t_j)$$

⇝ **Mixed strategies** – expected payoff to player 1:

$$\pi_1(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{k} \sum_{j=1}^{l} u_1(s_i, t_j) p_i q_j$$

Theorem 1. *There always exist mixed strategies* $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ *such that*

$$\pi_1(\boldsymbol{p}^*, \boldsymbol{q}^*) = \max_{\boldsymbol{p}} \min_{\boldsymbol{q}} \pi_1(s_i, t_j) \leq \min_{\boldsymbol{q}} \max_{\boldsymbol{p}} \pi_1(s_i, t_j)$$

2.8 GAME THEORY = MATHEMATICAL DISCIPLINE

1944 ***Theory of Games and Economic Behavior***
John von Neumann (1903 – 1957) & Oskar Morgenstern (1902 – 1976)

- **Application possibilities of game theory** – detailed formulation of economical problem
- **Axiomatic utility theory**
- **General formal description of a game of strat.**
- **2-player antagonistic finite games**
- ***n*-player cooperative games** (with transferable payoffs)
  ⇝ **von Neumann-Morgenstern's solution**
  (it is not unique, does not necessarily exist)

...

⇝ **Massive development of game theory and its applications**

The next step: Non-constant sum noncooperative games, cooperative games without transferable payoff

$$\begin{pmatrix} (3,-3) & \boxed{(\mathbf{2},\mathbf{-2})} \\ (0,0) & (1,-1) \end{pmatrix}$$

$$\begin{pmatrix} (3,3) & \boxed{(\mathbf{2},\mathbf{4})} \\ (0,6) & (1,5) \end{pmatrix} \rightsquigarrow \begin{pmatrix} (3,3) & \rightarrow & \boxed{(\mathbf{2},\mathbf{4})} \\ \uparrow & & \downarrow \\ (0,2) & \rightarrow & \underline{\underline{(\mathbf{4},\mathbf{5})}} \end{pmatrix}$$

$(\mathbf{4}, \mathbf{5})$ **... mutually best replies – equilibrium point**

2.9 JOHN FORBES NASH (*1928)

Equilibrium Concept

1949 ***Non-Cooperative Games*** (thesis; Ph.D. 1950)

- **Nash equilibrium** – introduction of the concept
- **Nash theorem** – proof of its existence

  Motivations:

  – Rational considerations of payoffs
  – Interactive adjustment processes, in which boudedly rational agents observe the strategies played by their likely opponents over time, and gradually learn to adjust their own strategies to earn higher payoffs, will eventually converge to a Nash equilibrium – if their converge to anything at all (only in the thesis)

  $\rightsquigarrow$ **confirmed by later experimental works.**

1950 ***Equilibrium Points in n-Person Games***
Proceedings of Nat. Acad. of Sciences of USA

1951 ***Non-Cooperative Games,*** Annals of Math.

Axiomatic Bargaining Theory

1950 ***The Bargaining Problem,*** Econometrica 18

1949 ***Two-Person Cooperative Games,*** ibid. 21

- Non-transferable payoffs
- The approach to cooperative games by their reduction to non-cooperative
- Axioms that a solution shall satisfy
- The proof of the existence of the unique solution: **Nash bargaining solution**