GROUP DECISION MAKING

How to join the preferences of individuals into the choice of the whole society

Main subject of interest: **elections = pillar of democracy**

Examples of voting methods

Consider n alternatives, from which one winning alternative shall be chosen.

Simple majority

Each voter chooses his most preferred alternative

$\longrightarrow$ the alternative with the greatest number of votes wins

Ostrakismos (Klesthenes, 508 př.kr.)

Assemblage: each citizen wrote on an **ostrakon** the name of the person who is, according to his opinion, dangerous for the freedom of citizens $\longrightarrow$ the person against whom the majority of votes was, had to leave Athens for 10 years (did not lose his citizen rights nor his property)

Ramon Llull (1235 – 1316)

The winner should also win in a duel with any of the remaining candidates.

Blanquerna (1282 – 1287),
Artifitium electiones pesonarum, (před 1283),
De arte electionis (1299)

Nicolaus Cusanus (1401 – 1464)

Llul winner does not necessarily exist.

Cusanus method, 1433: each voter gives 1 point to his least preferred candidate, 2 points to the one before the last, . . . , n points to his most preferred candidate. The winner is the candidate with the highest score.

De concordantia catholica, 1433

Marie Jean Antoine Nicolas Caritat, marquis de Condorcet (1743 – 1794)

***Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*, 1785**

Voting method: each voter strictly orders all alternatives according to his preferences $\longrightarrow$ **„Condorcet winner“:** such an alternative X that for any other alternative Y, the number of voters preferring X to Y is greater than the number of voters preferring Y to X.

☞ ***Example***. *The majority preferrs C to B, C to A, B to A.*

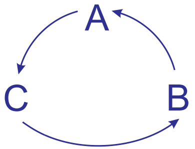
B, C win against A; majority prefers C to B, the winner is C.

Condorcet winner does not necessarily exist

☞ ***Example***. *Consider three different voters with the following preferences:*

		Voter		
		X	Y	Z
Preferences ranking	1.	A	C	B
	2.	B	A	C
	3.	C	B	A

Cycle: $A \succ B,\ B \succ C,\ C \succ A$



$\Longrightarrow$ it is always possible to find such an alternative that is able to beat the previous winner out on the basis of majority criterion (in a binary competition).

Condorcet paradox: Although all individual preferences are transitive, ”social preferences” obtained by majority elections are not necessarily transitive.

In democracy, in general, no best preference exists that could be possible to consider as a final, winning and the most equitable.

Jean Charles de Borda (1733 – 1799)

***Mémoire sur les élections au scrutin*, 1781 (1770)**

Critique of election methods in Paris Academy: the winner of a majority election can be the generally least preferred candidate.

☛ ***Example***: *12 voters with the following preferences choose one of three candidates, A, B, C.*

Number of voters

		5	4	3
Pořadí preferencí	1.	Alice	Barbara	Cecil
	2.	Cecil	Cecil	Barbara
	3.	Barbara	Alice	Alice

Majority method: ***Alice*** $\succ$ *Barbara* $\succ$ *Cecil* (5 : 4 : 3)

$\longrightarrow$ **Discord:** decision between Alice and Cecil:

Cecil $\succ$ *Alice* (7 : 5)

Borda method: Each alternative obtains from each voter the number of points according to its position at his preference ordering: if it is at the last place, it gains 1 point, at the one before last, 2 points, . . . , at the first place n points, where n is the number of alternatives. The winner is the alternative with the highest score.

Used in Paris Academy: 1796 – 1803

The winner is not necessarily at the first place for any of voters; no cycles – only draws

Score:

		Number of voters		
		5	4	3
Preference ranking	1.	Alice	Barbara	Cecil
	2.	Cecil	Cecil	Barbara
	3.	Barbara	Alice	Alice

Alice ... $\mathbf{3 \cdot 5 + 1 \cdot 4 + 1 \cdot 3 = 22}$

Barbara ... $\mathbf{1 \cdot 5 + 3 \cdot 4 + 2 \cdot 3 = 23}$

Cecil ... $\mathbf{2 \cdot 5 + 2 \cdot 4 + 3 \cdot 3 = 27}$ ... **Borda winner**

Pairwise decision: *Barbara* $\succ$ *Alice* (7 : 5)

Cecil $\succ$ *Barbara* (**8** : 4)

Cecil $\succ$ *Alice* (**7** : 5)

$\longrightarrow$ Cecil wins in the comparison with the remaining candidates

Charles Lutwidge Dodgson (1832 – 1898)

***A Method of Taking Votes on More than Two Issues*, 1876**

A voting system inspired by Condorcet: the winner is the candidate who is ”closest” to the Condorcet winner: he would became Condorcet winner by the lowest number of changes in preferences of voters

Duncan Black (1908 – 1991)

- ***On the Rationale of Group Decision Making*, 1948**
- ***Theory of Committees and Elections*, 1958**

Propagated the works of Charles Dodgson

Black method:

$$\text{the winner} = \begin{cases} \text{Condorcet winner, if he exists} \\ \\ \text{Borda winner otherwise} \end{cases}$$

Kenneth Arrow (*1921)

Social Choice and Individual Values, 1951

- $A = \{x, y, \ldots, z\}$ – **set of alternatives**
- $Q = \{1, 2, \ldots, n\}$ – **set of individuals, society**
- For each individual $i \in Q$ and any alternatives x and y, a **complete ordering (**i **prefers** x **to** y **or is indifferent between** x, y**)** is defined:
  - $x \succ_i y$ iff it is not $y \succeq_i x$ (**i prefers x to y**),
  - $x \approx_i y$ iff $x \succeq_i y \wedge y \succeq_i x$ (**i is indiff. between x, y**).
- Further, relations **preference** $\succ_i$ and **indifference** $\approx_i$ are defined:
  - $x \succ_i y$ iff it is not $y \succeq_i x$ (**i prefers x to y**),
  - $x \approx_i y$ iff $x \succeq_i y \wedge y \succeq_i x$ (**i is indiff. between x, y**).

Schematic representation of preferences:

for example:

R^1	R^2	R^3
x	y	$x - y$
y	x	

Denote the set of all orderings of alternatives

$$\mathcal{R} = \{R^1, R^2, \ldots, R^m\}$$

Definition. By a **profile of preference orderings** for the individuals of the society we mean an n-tuple of orderings, $(R_1, \ldots, R_n)$, where R_i is the preference ordering for the ith individual.

Definition. By a **social welfare function** we mean a rule which associates to each profile of preference orderings a preference ordering for the society itself, i.e.

$$F : \mathcal{R}^n \longrightarrow \mathcal{R};\ (R^1, R^2, \ldots, R^n) \mapsto R;$$

we will write simply $x \succeq y$ instead of $(x, y) \in R$

(society prefers x to y or is indifferent between x, y); analogically for $x \succ y$, $x \approx y$.

Condition 1

- *The social welfare function* $\boldsymbol{F}$ *is defined for all possible profiles of individual orderings.*
- *The number of alternatives in* $\boldsymbol{A}$ *is greater than or equal to three, i.e.* $|\boldsymbol{\mathcal{A}}| \geq \mathbf{3}$.
- *There are at least two individuals, i.e.* $|\boldsymbol{Q}| \geq \mathbf{2}$.

Remark:

Does not hold e.g. for Condorcet method

Condition 2 (positive association of social and individual values).

If the welfare function asserts that x is preferred to y for a given profile of individual preferences, it shall assert the same when the profile is modified as follows:

- *The individual paired comparisons between alternatives other than x are not changed*
- *Each individual paired comparison between x and any other alternative either remains unchanged or it is modified in x's favor.*

Remark:

Consider the following preference profiles.
If F is such, that for a profile

R^1	R^2	R^3
x	$x - z$	$x - y - z$
y	y	
z		

it is $y \succ z$, then it would contradict the intuition if for a profile

R^1	R^2	R^3
$x - y$	$x - z$	y
z	y	$x - z$

it would not remain $y \succ z$:

Condition 3 (independence of irrelevant alternatives).

Let B be any subset of alternatives in A. If a profile of orderings is modified in such a manner that each individual's paired comparisons among the alternatives of B are left invariant, the social orderings resulting from the original and modified profiles of individual orderings should be identical for the alternatives in B.

Remark:

Condition 3 does not hold for a Borda method:

	V	W	X	Y	Z
1.	**A**	**A**	B	B	C
2.	B	C	C	C	B
3.	C	B	**D**	**D**	**D**
4.	**D**	**D**	**A**	**A**	**A**

Score: A . . . 11
D . . . 8

Preference of society: A $\succ$ D

Restriction of the set of alternatives to $\{$A, D$\}$ **:**

	V	W	X	Y	Z
1.	**A**	**A**	**D**	**D**	**D**
2.	**D**	**D**	**A**	**A**	**A**

Počet bodů: A . . . 7
D . . . 8

Preference of society: D $\succ$ A

Condition 4 (citizen's sovereignity).

For each pair of alternatives x and y there is some profile if individual orderings such that society prefers x to y.

Condition 5 (non-dictatorship).

There is no individual with the property that whenever he prefers x to y (for any x and y) society does likewise, regardless of the preferences of other individuals.

Condition 4 (citizen's sovereignity).

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Theorem (Arrow's Impossibility Theorem).

The conditions 1, 2, 3, 4 and 5 are **inconsistent***.*

It means that there does not exist any welfare function which possesses the properties demanded by these conditions.

In other words, if a welfare function satisfies conditions 1, 2 and 3, then it is either imposed or dictatorial.

Proof of Arrow's Impossibility Theorem

1. Suppose that $V \neq \emptyset$ is a **minimal decisive set**, i.e. there exist alternatives $x, y \in A$, such that V is decisive for (x, y), but no proper subset $V' \subset V$ is decisive for any ordered pair of alternatives.

V exists:

- Q is decisive for any pair of alternatives (so-called **Pareto optimality**; follows from conditions 1–4).
- Individuals can be removed one at a time until the remaining set is no longer decisive for any pair. Then, if $V = \emptyset$, then a pair (x, y) would exist, such that $\emptyset$ would be a decisive set; but in this case $Q = Q \setminus \emptyset$ would not be decisive for (x, y), which is a contradiction.

2. Choose an arbitrary $j \in V$; denote $W = V \setminus \{j\}$, $U = Q \setminus V$ (since $|Q| \geq 2$, at least one of the sets U, W is non-empty). Choose an arbitrary $z \in \mathcal{A}$, $z \neq x, y$. Consider the following profile:

$\{j\}$	W	U
x	z	y
y	x	z
z	y	x

- For all $i \in V = W \cup \{j\}$ it is $x \succ_i y$, hence $x \succ y$.
- It must be also $y \succeq z$ (otherwise W would be decisive for (z, y), which is a contradiction with the minimality of V).
- From transitivity we have: $x \succ z$.
- But j is the only individual, who prefers x to z; since V is minimal, $\{j\}$ can not be a proper subset of V, hence $V = \{j\}$.

3. By now, we have shown that for every $z \neq x$, $\{j\}$ is decisive for (x, z). Now consider any $w \in \mathcal{A}$, $w \neq x, z$. We will show that $\{j\}$ is also decisive for (w, z) and (w, x). Consider the following profiles:

$\{j\}$	U
w	z
x	w
z	x

From Pareto optimality, we have: $w \succ x$;
$\{j\}$ is decisive for (x, z), hence $x \succ z$;
from transitivity: $w \succ z$, tj.
$\{j\}$ is decisive for (w, z).

$\{j\}$	U
w	z
z	x
x	w

$\{j\}$ is decisive for (w, z), thus $w \succ z$;
from Pareto optimality: $z \succ x$;
from transitivity: $w \succ x$, i.e.
$\{j\}$ is decisive for (w, x).

We have therefore shown that $\{j\}$ is decisive for **any pair of alternatives** – thus it is a **dictator** from the condition 5.

Remark. Simple majority principle is the only one satisfying the following conditions:

- **Decisiveness:** For any profile of individual choices, it specifies a unique group decision for each paired comparison.
- **Anonymity:** It does not depend upon the labeling of individuals.
- **Neutrality:** It does not depend upon the labeling of the two alternatives.
- **Positive responsiveness:** If for a given profile the rule specifies that $x \succeq y$ and if a single individual then changes his paired comparison in favor of x, while the remainder of the society maintain their former choices, then the rule requires that in the group decision it is $x \succ y$.

Denote $N_x = |\{i \in Q; x \succeq_i y\}|$, $N_y = |\{i \in Q; y \succeq_i x\}|$,

$$N_I = |\{i \in Q; x \approx_i y\}|.$$

- **Anonymity:** group decision upon x, y depends only upon N_x, N_y, N_I,
- from **neutrality** it follows: $x \approx y$, iff $N_x = N_y$,
- by a repeated use of a **positive responsiveness** it is possible to show:
  $x \succ y$ if and only if $N_x > N_y$, resp.
  $y \succ x$ if and only if $N_y > N_x$.