EXPLICIT AND NORMAL FORM GAMES

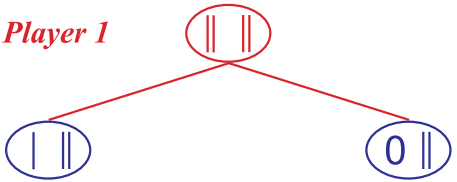
EXPLICIT FORM GAMES

☞ ***Example***. **Game of Nim**

Consider a simple game where two players – let us denote them 1, 2 – have two piles at the table in front of them, each consisting of two beans. Player 1 has to take one or two beans away from one pile (the beans can not be returned back). Then there is a second player's turn: he has to take one or two beans from one pile, too. In this way the players take turns, until one of them takes away the last bean – and this player loses.

Provided you could choose whether you have the first or the second turn, what would you decide for?

The game can be represented by the model named **explicit form game** or **game tree**.


Player 1
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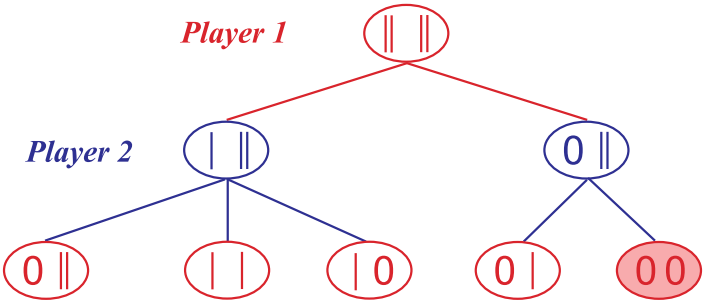
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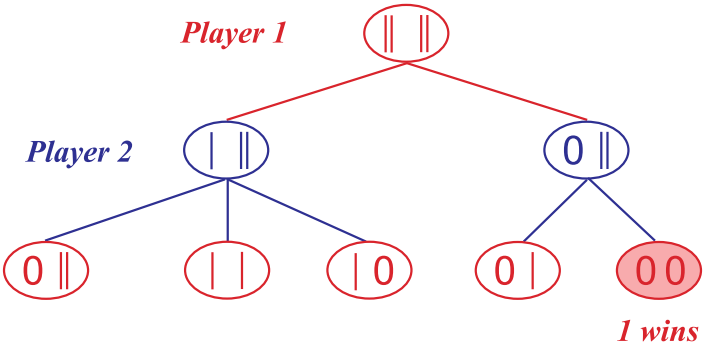
Player 1
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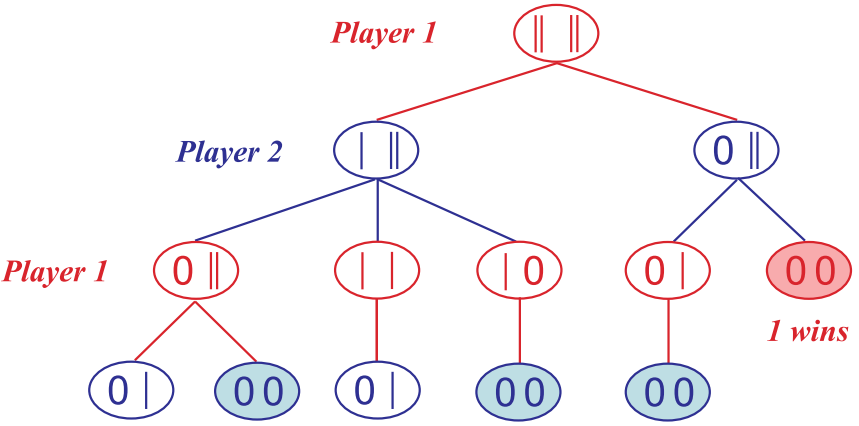
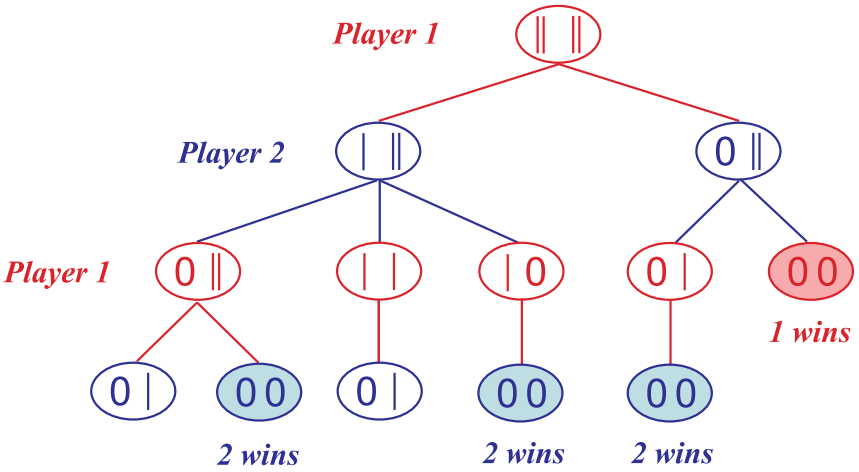
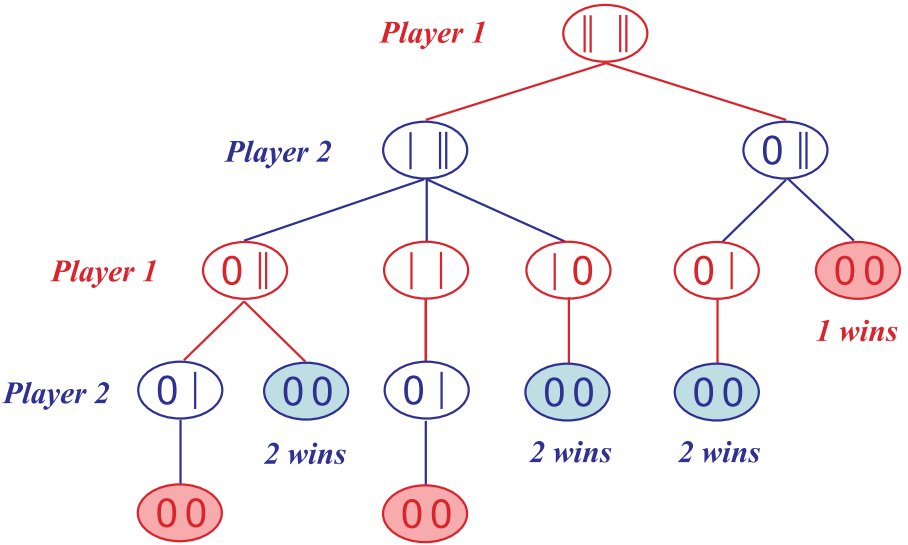
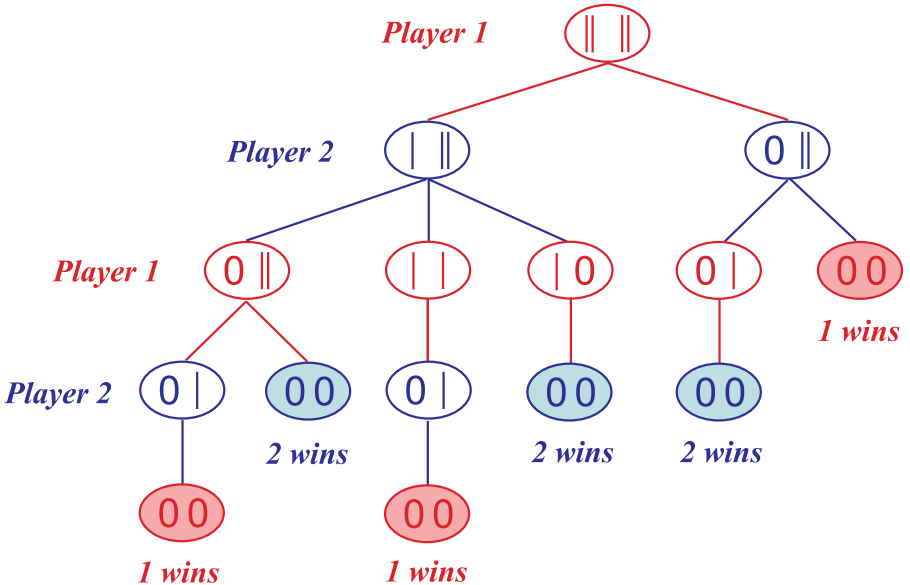
1 wins


Player 1
Player 2
Player 1
1 wins


Player 1
Player 2
Player 1
1 wins
2 wins
2 wins
2 wins


Player 1
Player 2
Player 1
Player 2
1 wins
2 wins
2 wins
2 wins


Player 1
Player 2
Player 1
Player 2
1 wins
2 wins
2 wins
2 wins
1 wins
1 wins

This model shows all situations that can occur in the game. To each situation one **node** corresponds, from each node a certain number of **edges** comes out, that correspond to possible decisions, so called **turns** of a given player. If a player decides for some turn, then he induces a new situation in which the second player decides – to this new situation another node corresponds, that is connected with the previous one by an edge.
Drawing the tree, we usually proceed from the top downwards or from left to right, alternating the first player's and second player's nodes regularly.
There is always just one node with the property that no edge enters it; such node is called an **initial node** or **root** of the tree. Further, there are nodes from which no edges go out; these nodes are called **terminal nodes** and they correspond to positions in which the result of the game is clear and the game ends.

From the figure it is obvious that whatever strategy the first player chooses, the second player can choose a strategy that leads him to the victory.

☛ *Example*. Game of Nim – Modification

In the game from the previous example , consider three piles instead of two, each consisting of two beans again; the rules of the game are the same. Which player has a winning strategy?

☞ *Example*. Game of Nim – Modification

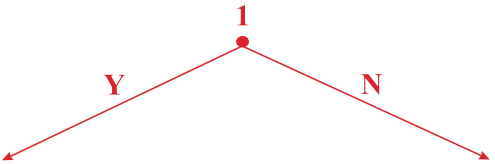
In the game from the previous example, consider three piles instead of two, each consisting of two beans again; the rules of the game are the same. Which player has a winning strategy?

Hint: The first player can take one pile away from the table and hence force the opponent to the position of the first player in the previous variant with two piles.

☞ *Example*. Voting on Wage Rise

Three legislators vote of their own wage rise. Each of them wishes the rise. Nevertheless, together with voting ”YES” a legislator faces up to a loss of voter’s favour worth c. The benefit b from the rise exceeds the loss c, $b > c$.

Provided the legislators vote successively and publicly, is it better to be first or last in the election? The last one can see what the situation is like and can possibly decide whether the rise passes or not. Is it therefore the most advantageous?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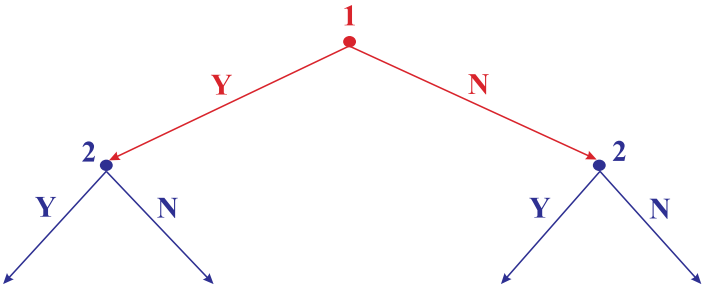
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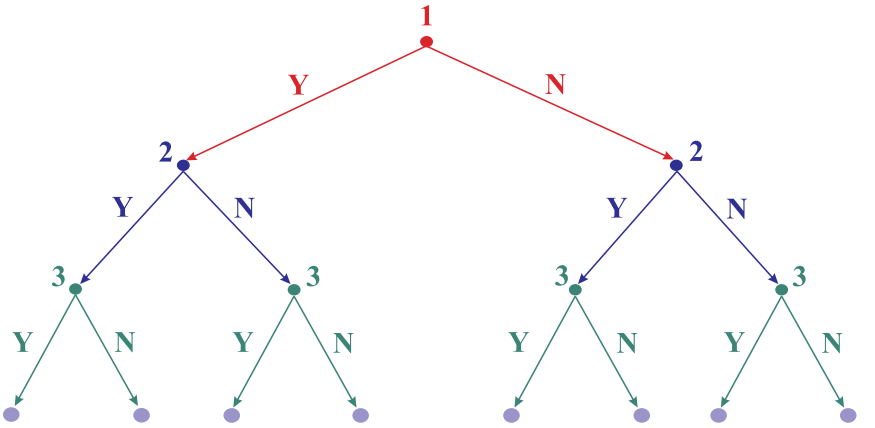
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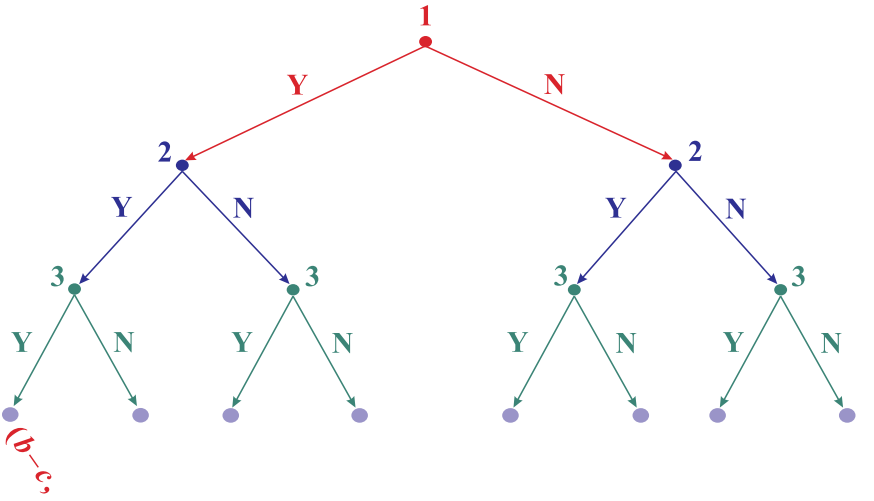
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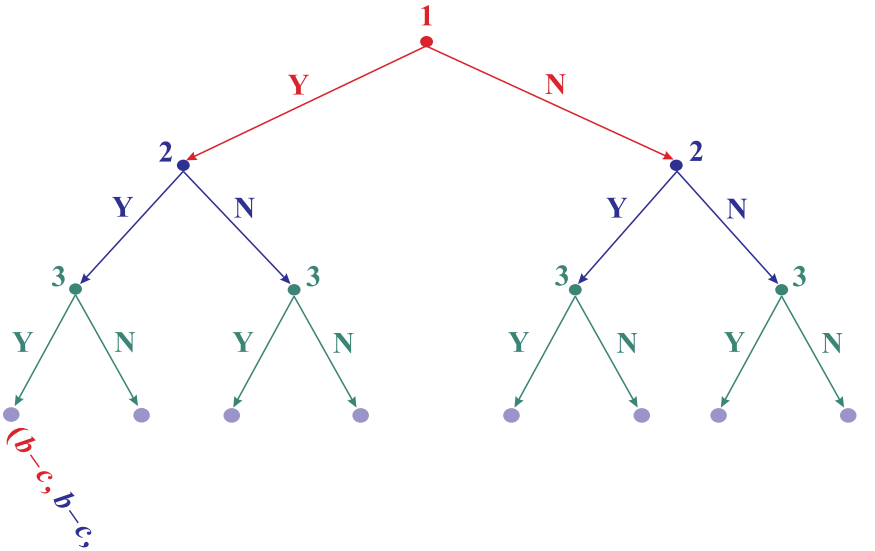
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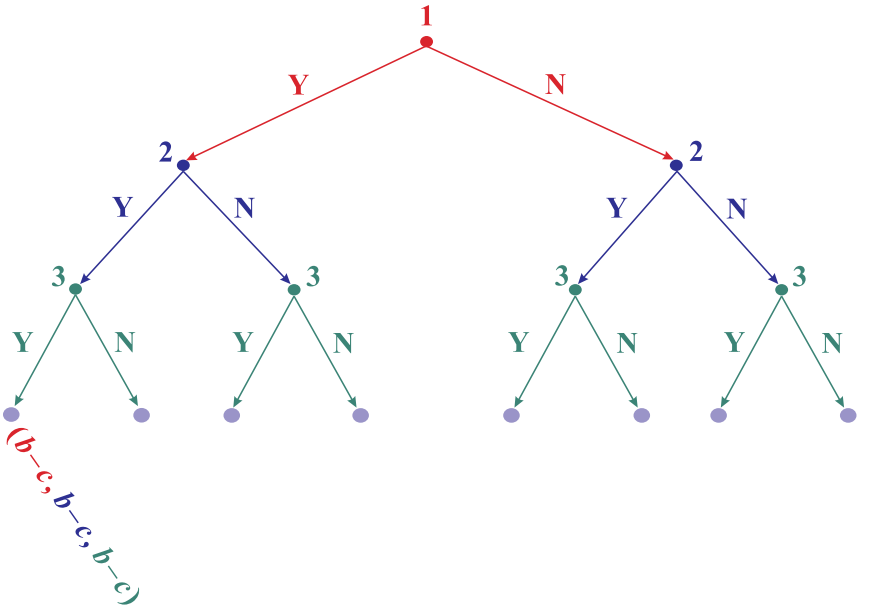
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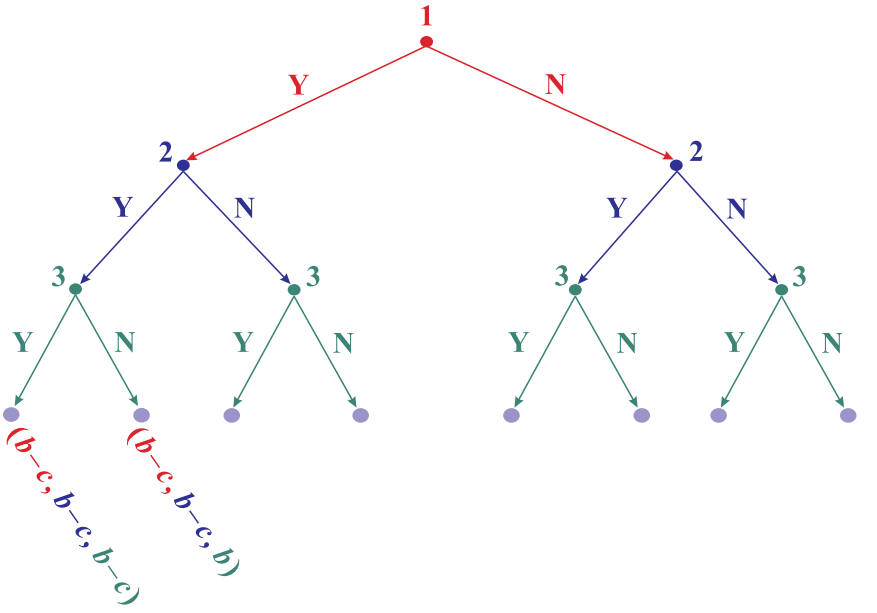
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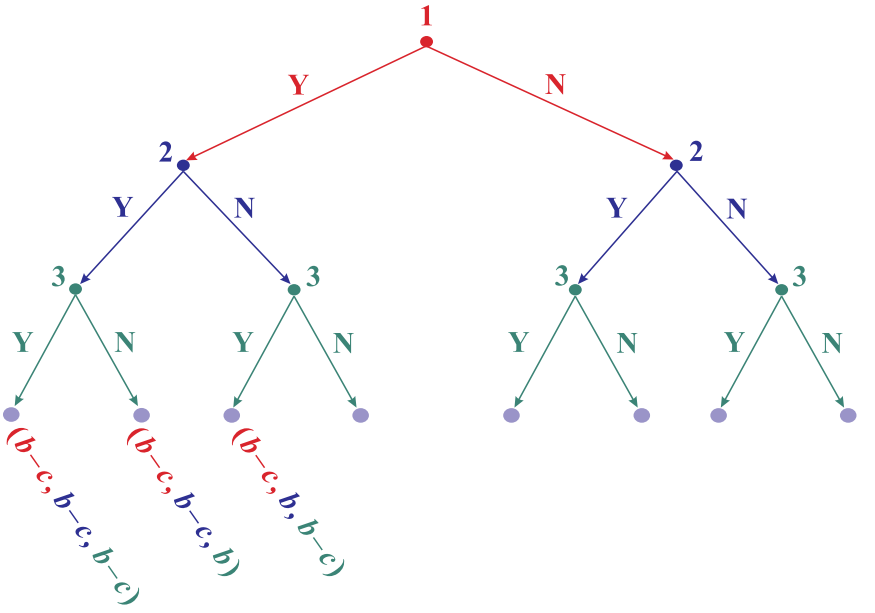
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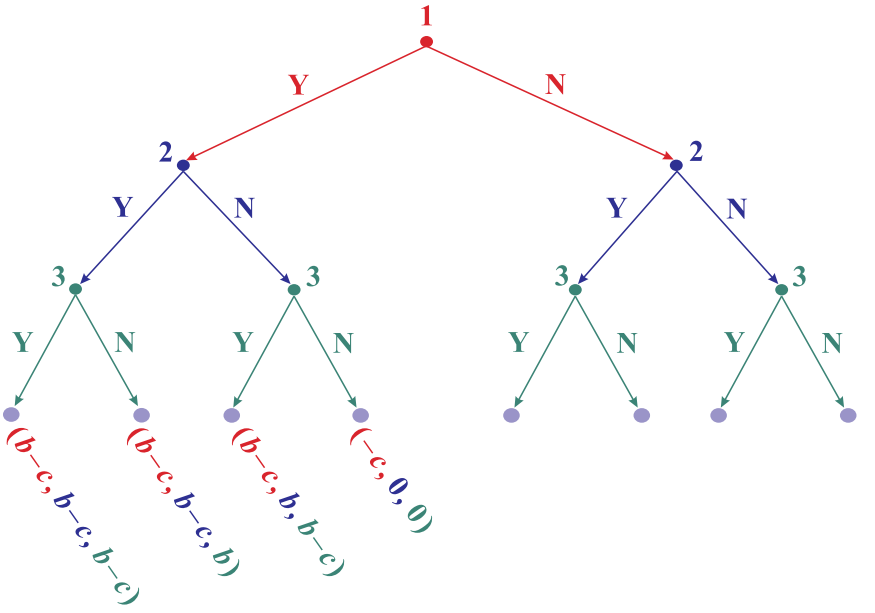
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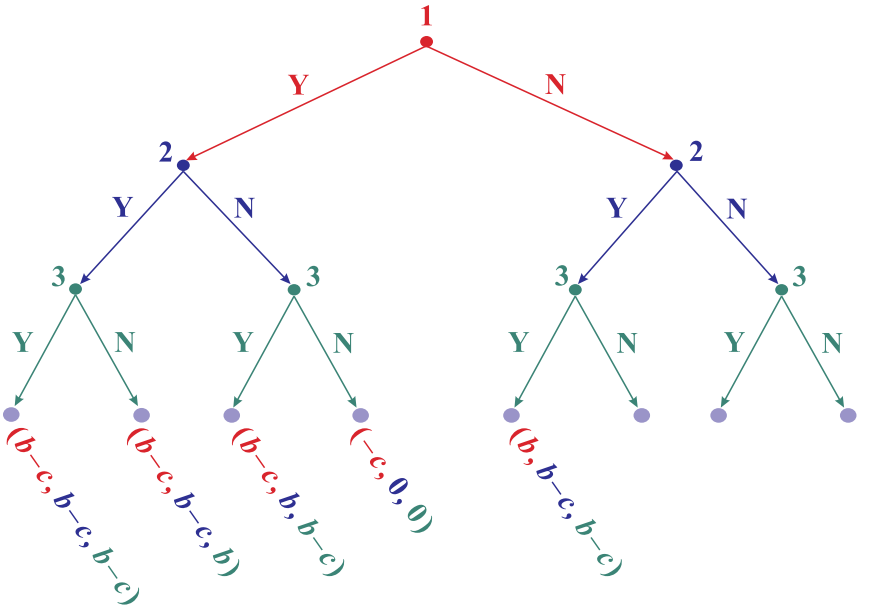
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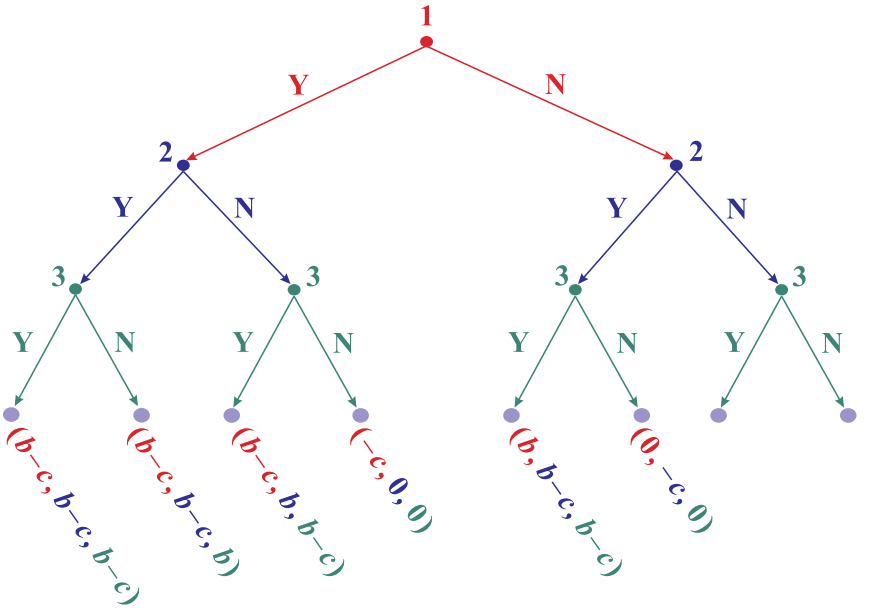
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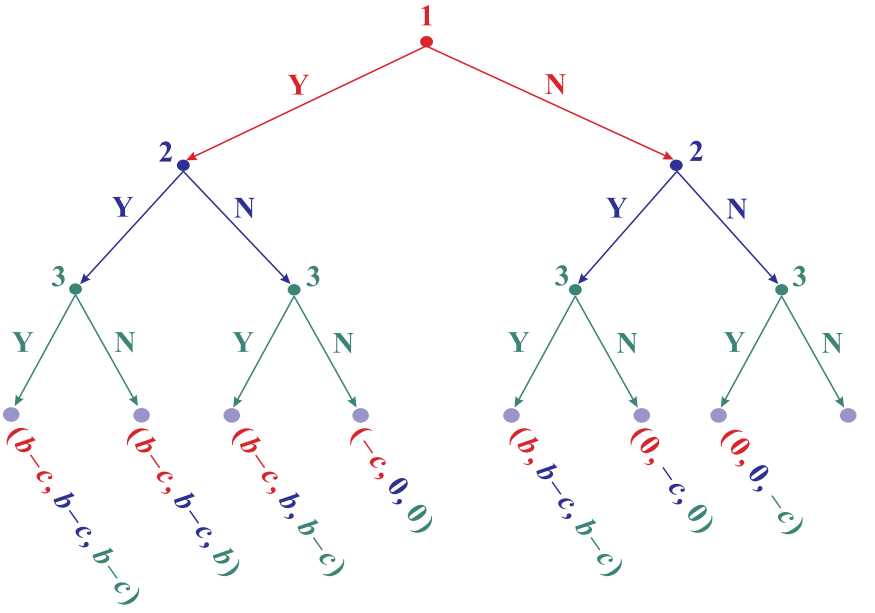
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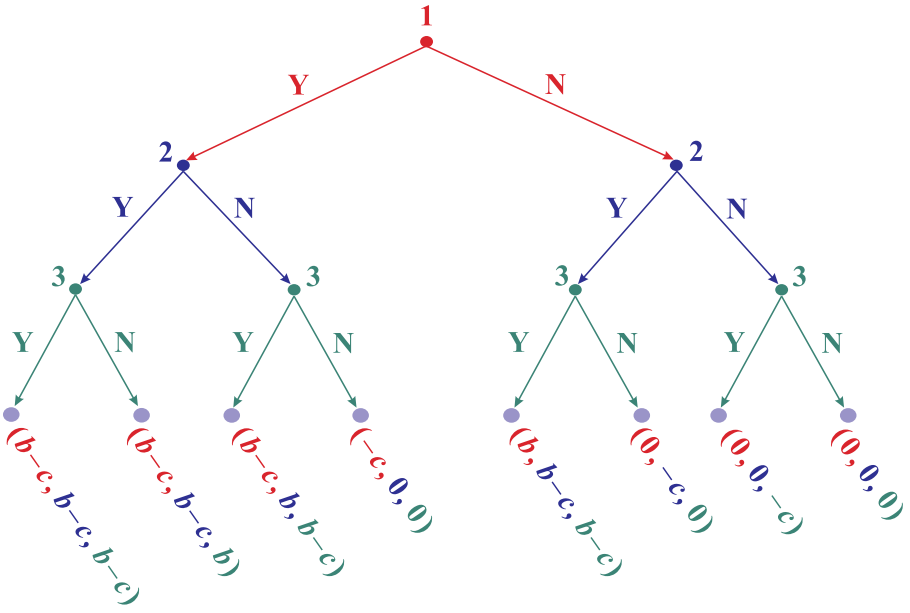
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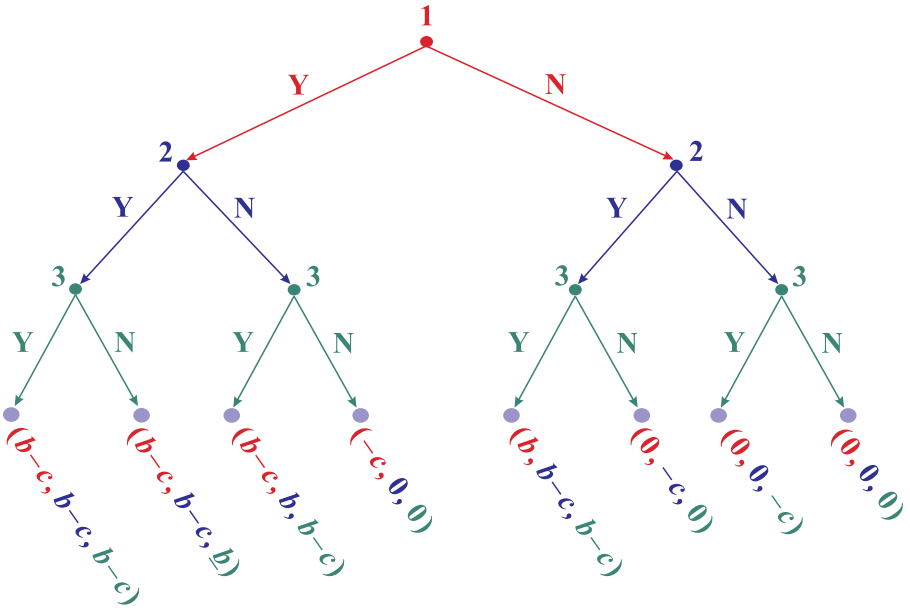
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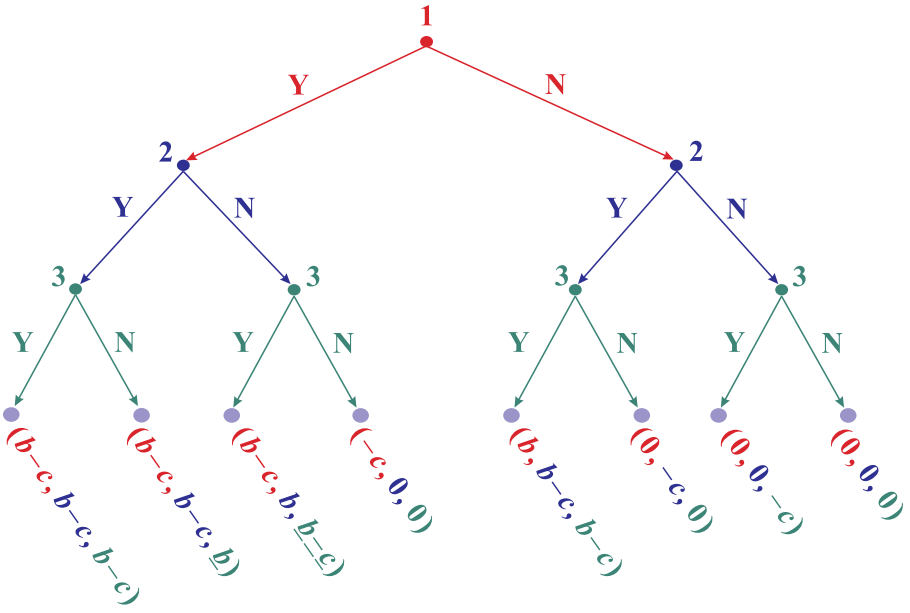
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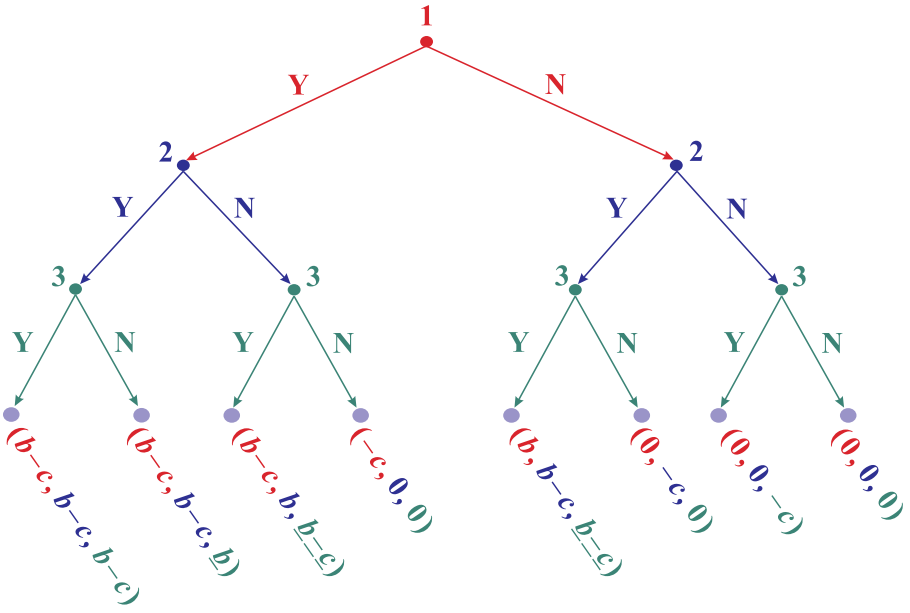
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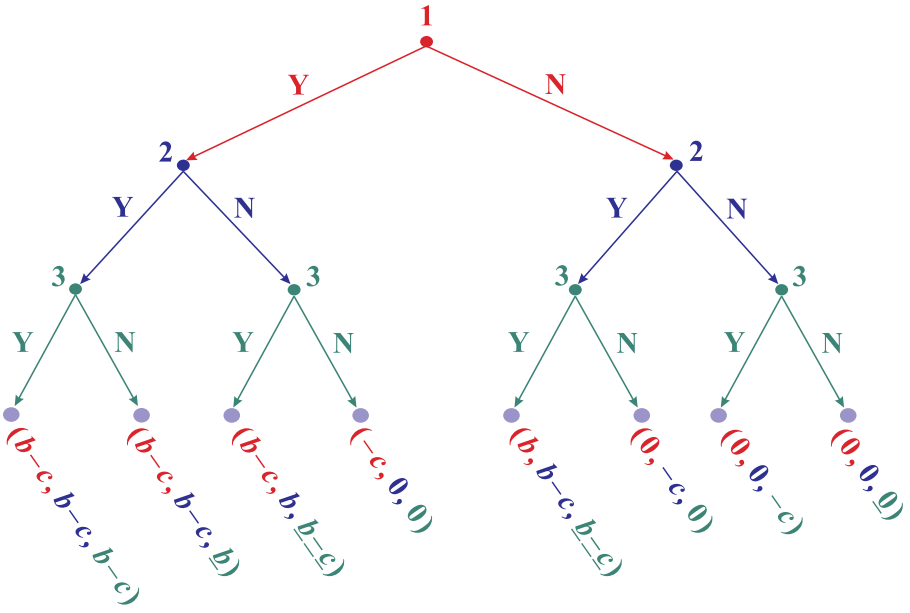
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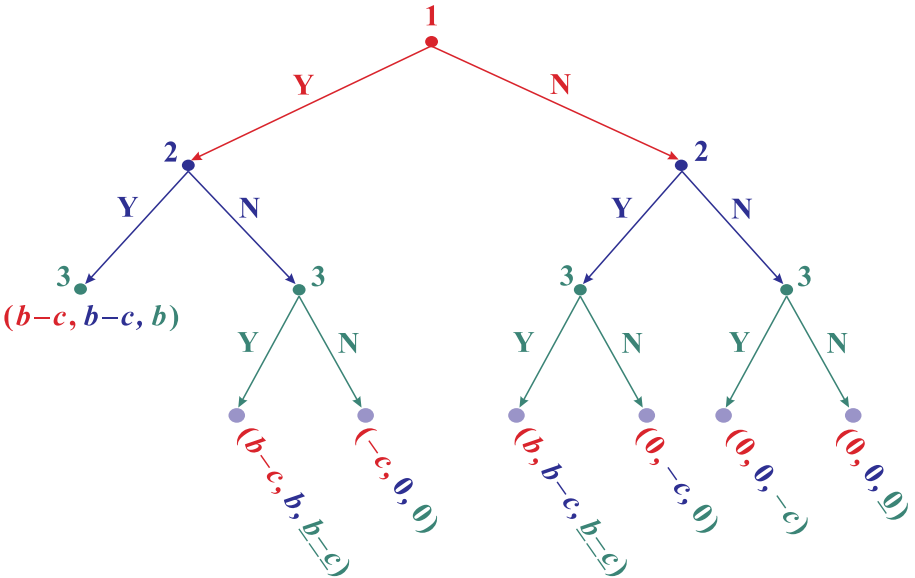
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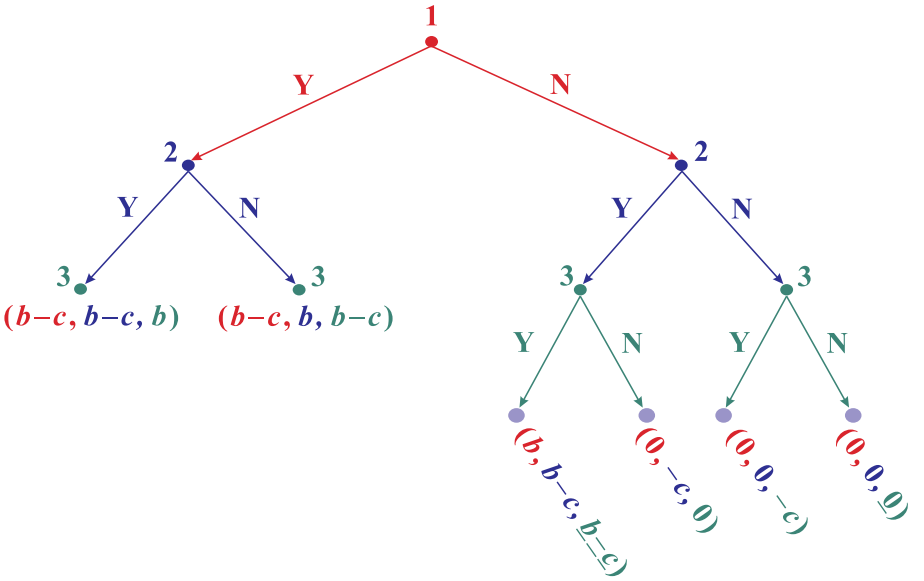
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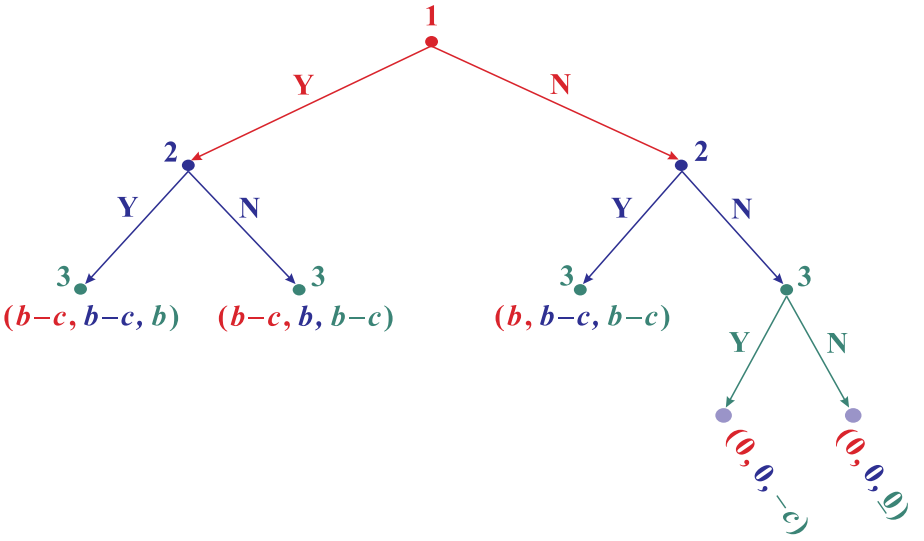
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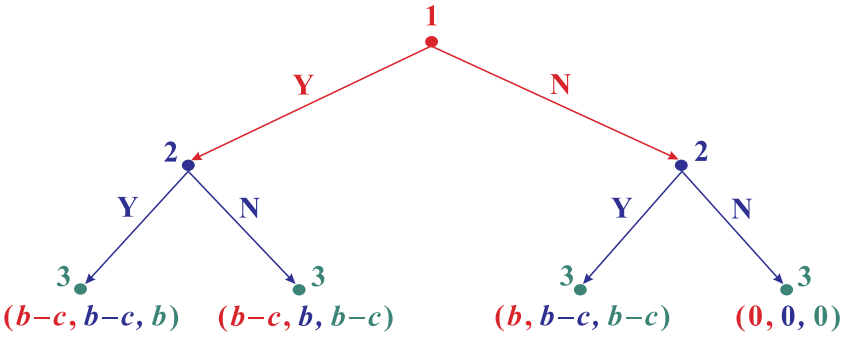
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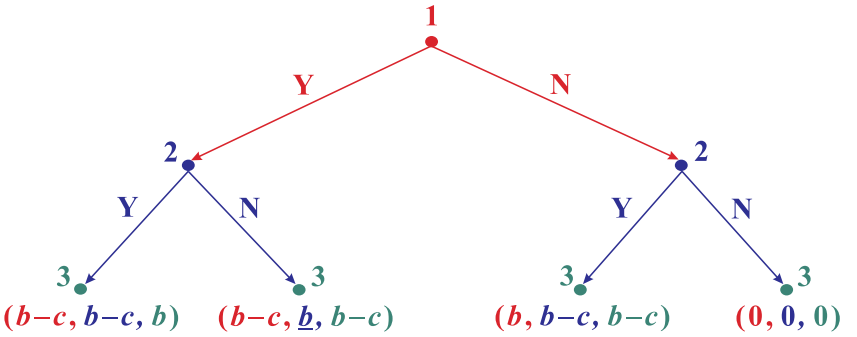
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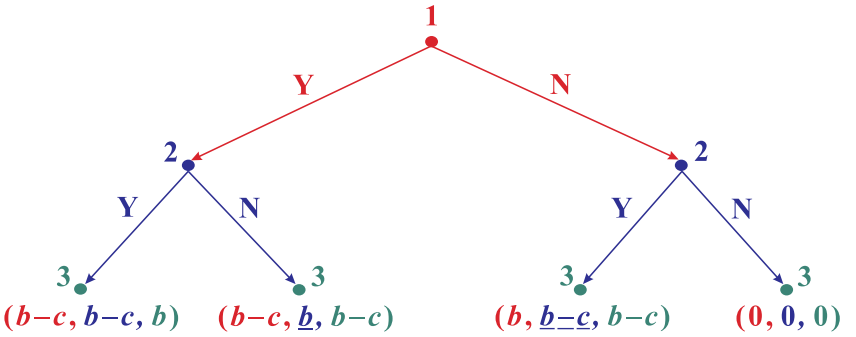
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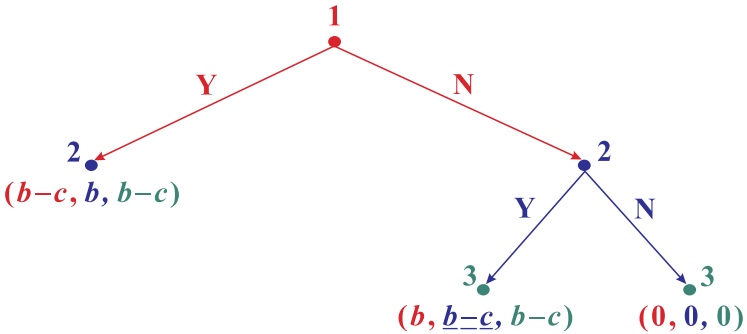
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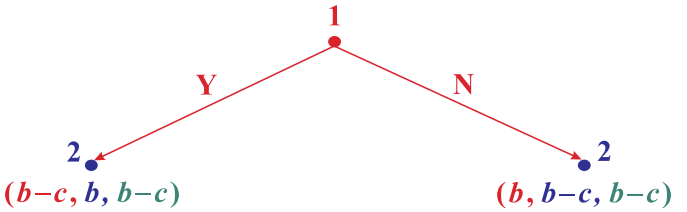
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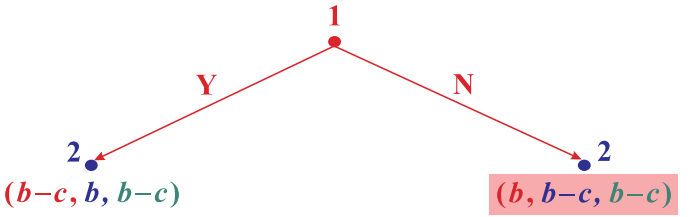
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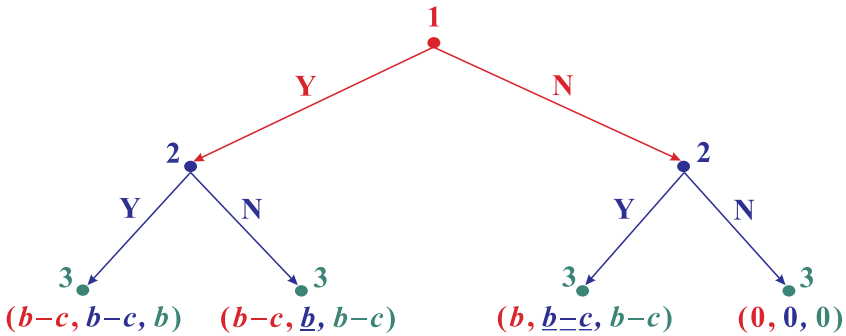
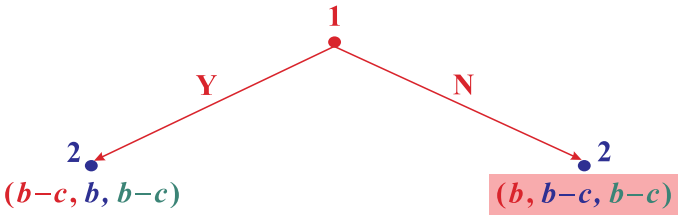
The number by a node expresses which legislator's turn it is. The triplet of numbers by each of the terminal nodes expresses the profit of the first, second and third legislator respectively.
Proceed in the tree from the bottom upwards. In the nodes with number 3 the third legislator decides, whose profit is given by the third number in the triplet. If a situation corresponding to the very left node with number 3 occurs, the third legislator decides between the third numbers in triplets $(b-c, b-c, b-c)$ and $(b-c, b-c, \underline{b})$; since $b > b-c$, it is clear that he chooses $(b-c, b-c, \underline{b})$. In the same way we can go through all the nodes with number 3 and label an outcome chosen by the third legislator (in the figure underlined).

The second legislator therefore chooses from the following alternatives in each of his nodes:



Profits of the second legislator are expressed by second numbers in triplets, more convenient alternatives are underlined.

The first legislator can consider the choices of his colleagues in particular situations in advance and he can see that, strictly speaking, he decides between two possibilities:



More advantageous is obviously the alternative on the right. Hence, if the first legislator votes ”NO”, the wages will rise anyway and the loss resulting from voting ”YES” is carried by the others.

The described reasoning is called **backward induction** – on the base of anticipating the future, the most convenient alternatives are deduced at the beginning of the decision.

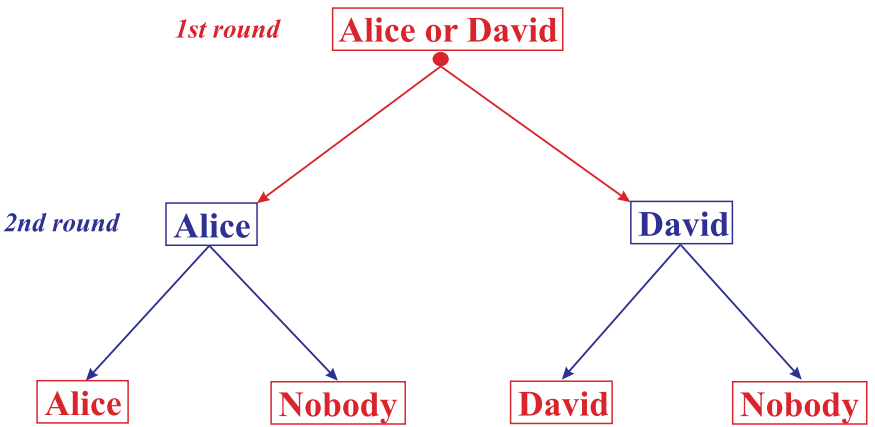
☞ *Example*. Two-Stages Committee Voting

Martin, Peter and Paul are the membership committee of the very exclusive Sharebroker Society. The final item on their agenda one morning is a proposal that Alice should be admitted as a new member. No mention is made of another possible candidate called David, and so an amendment to the final item is proposed. The amendment states that Alice's name should be replaced by David's. The rules for voting in committees call for amendments to be voted on in the order in which they are proposed. The committee therefore begins by voting on whether David should replace Alice. If Alice wins, they then vote on whether Alice or Nobody should be made a new member. If David wins, they then vote on whether David of Nobody should be made a new member. Preferences of particular members are the following:

Ranking	**Martin**	**Peter**	**Paul**
1.	Alice	Nobody	David
2.	Nobody	Alice	Alice
3.	David	David	Nobody

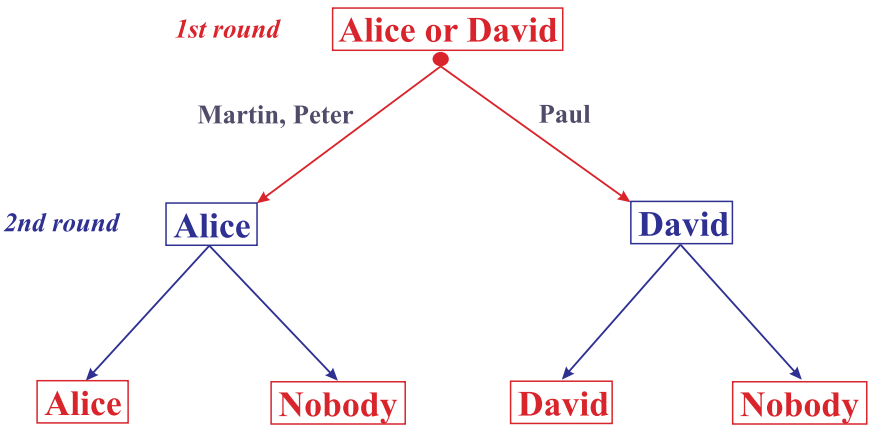
If everybody just voted according to their rankings, the election would go off in the following way: in a vote between Alice and David, Alice would win because both Martin and Peter rank Alice above David and so Paul would be outvoted. Thus, if there is no strategic voting, Alice will be elected to the club because she will also win when she is matched against Nobody.

Martin	Peter	Paul
Alice Nobody David	Nobody Alice David	David Alice Nobody



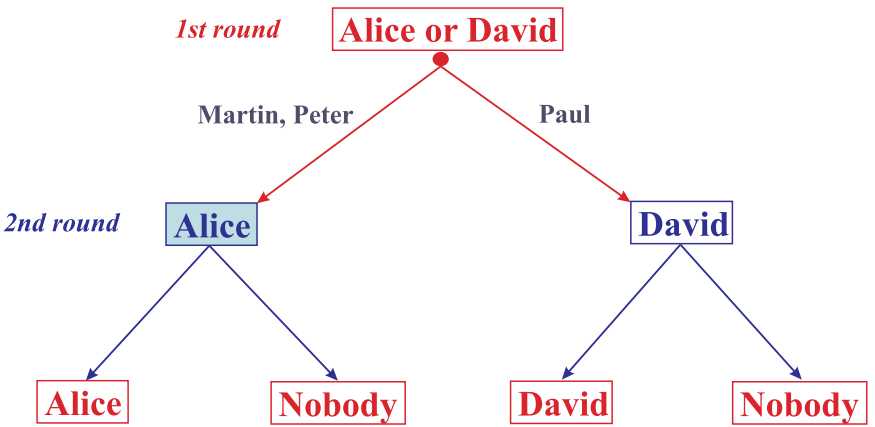
Sincere voting:

Martin	Peter	Paul
Alice	Nobody	David
Nobody	Alice	Alice
David	David	Nobody



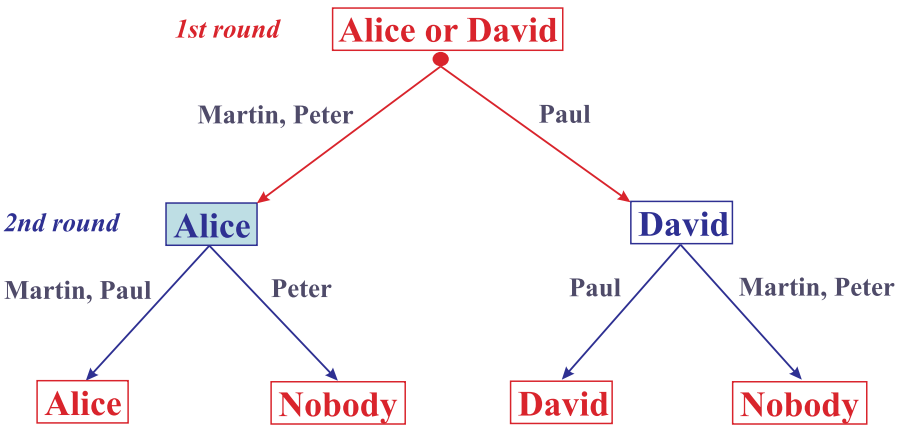
Sincere voting:

Martin	Peter	Paul
Alice	Nobody	David
Nobody	Alice	Alice
David	David	Nobody



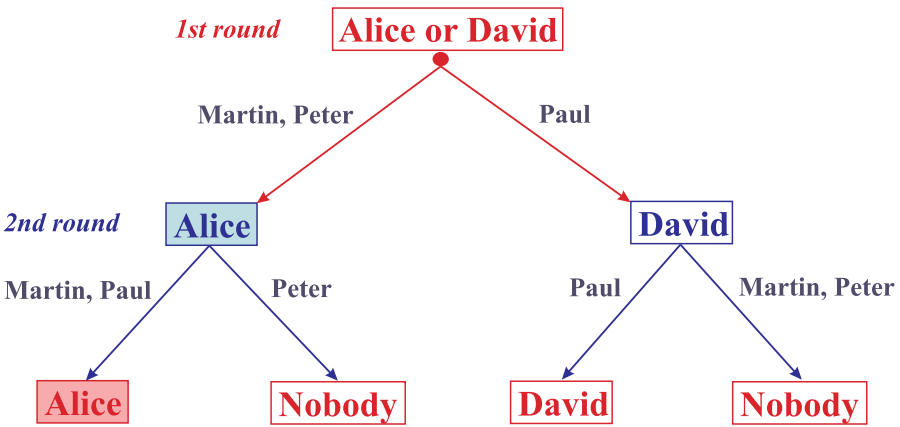
Sincere voting:

Martin	Peter	Paul
Alice	Nobody	David
Nobody	Alice	Alice
David	David	Nobody



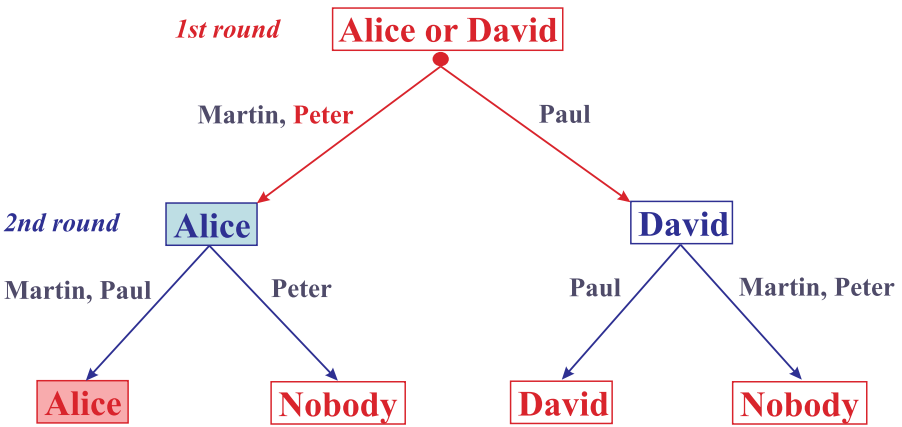
Sincere voting:

Martin	Peter	Paul
Alice	Nobody	David
Nobody	Alice	Alice
David	David	Nobody



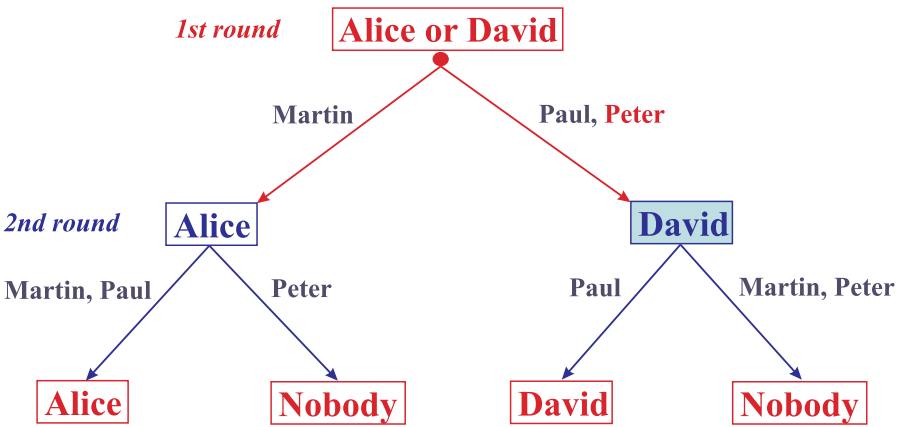
Strategic voting:

Martin	Peter	Paul
Alice	Nobody	David
Nobody	Alice	Alice
David	David	Nobody



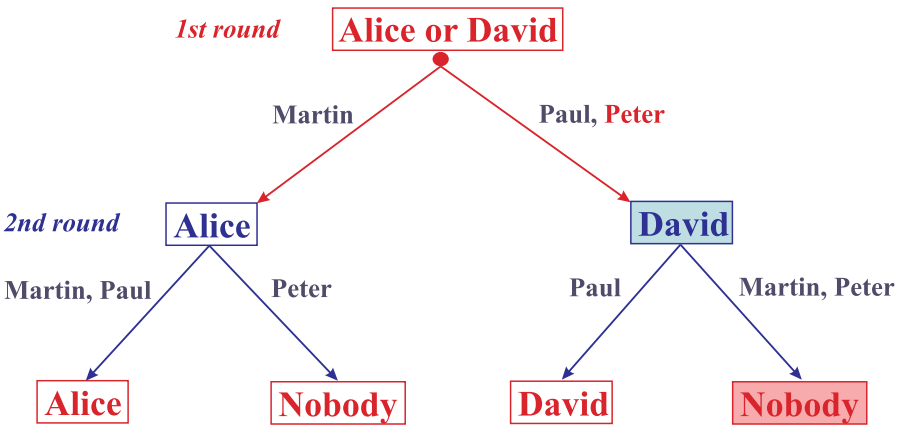
Strategic voting:

Martin	Peter	Paul
Alice	Nobody	David
Nobody	Alice	Alice
David	David	Nobody



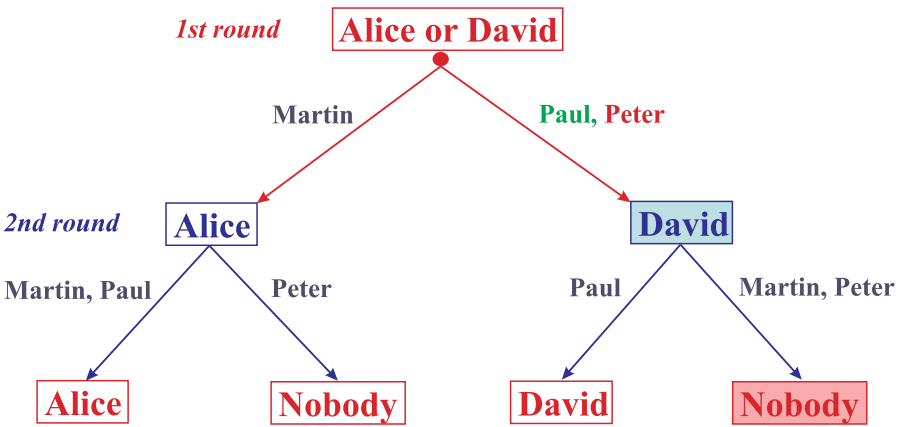
Strategic voting:

Martin	Peter	Paul
Alice	Nobody	David
Nobody	Alice	Alice
David	David	Nobody



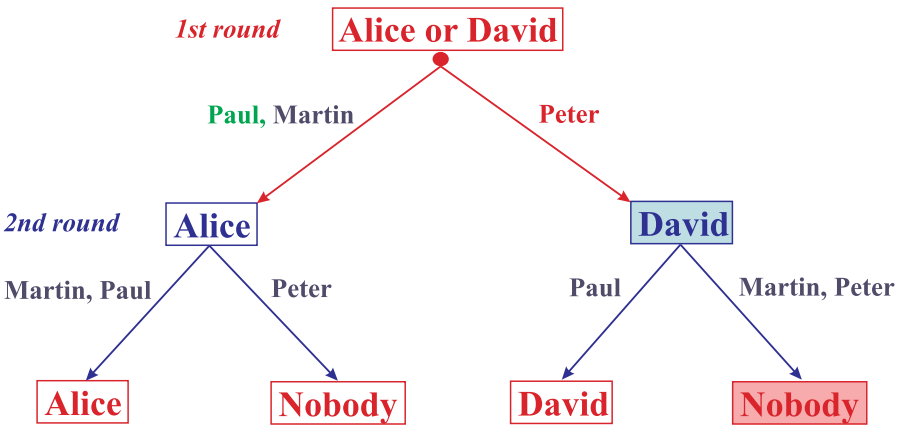
Strategic voting:

Martin	Peter	Paul
Alice	Nobody	David
Nobody	Alice	Alice
David	David	Nobody



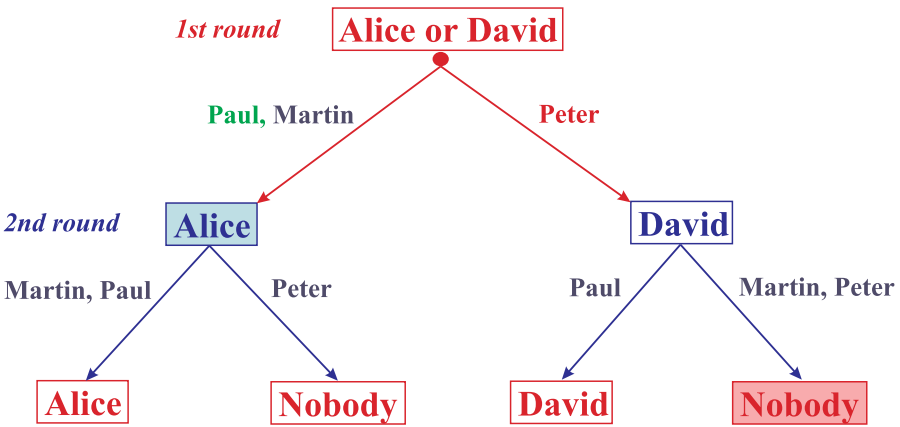
Strategic voting:

Martin	Peter	Paul
Alice	Nobody	David
Nobody	Alice	Alice
David	David	Nobody



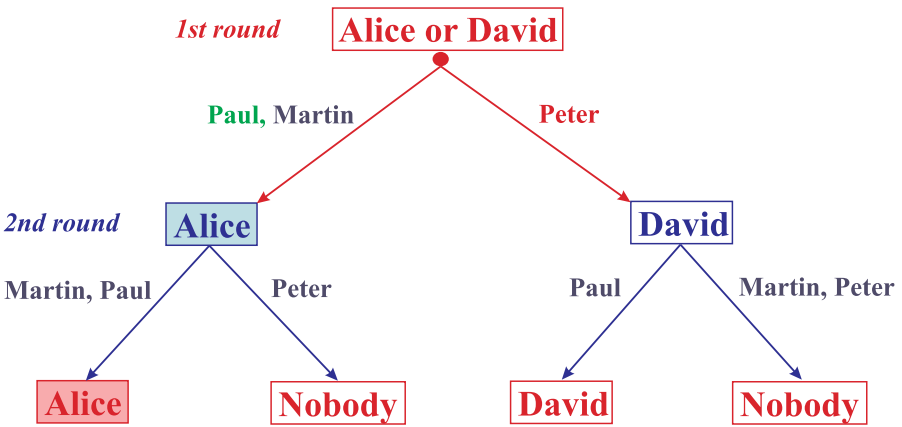
Strategic voting:

Martin	Peter	Paul
Alice	Nobody	David
Nobody	Alice	Alice
David	David	Nobody



Strategic voting:

Martin	Peter	Paul
Alice	Nobody	David
Nobody	Alice	Alice
David	David	Nobody



However, if Peter looks ahead, he will see that there is no point in voting against David at the first vote. If David wins the first vote, then Nobody will triumph at the second vote, and Nobody is Peter's first preference. Thus, Peter should switch his vote from Alice at the first vote, and cast his vote instead for David, who is the candidate he likes the least. If Paul and Martin do not also vote strategically, the result will be that Nobody is elected.

But Paul may anticipate that Peter will vote strategically and he can vote strategically, too, by switching his vote from David to Alice; he thereby ensures that Alice is elected rather than Nobody that is his least desired alternative.

☞ *Example*. Sophisticated Voting in Juridical Systems

Consider three juridical systems in which three judges decide:

1. **Status Quo** (used e.g. in the USA): First the guilt or innocence of the defendant is decided, then, in the case of the guilt, the punishment is decided.

2. **Roman Tradition**: After hearing the evidences, the judges vote downwards from the most severe sentence to the mildest one (possibly the release). For example, first they vote on whether to impose a death sentence or not; if not, whether to impose a life prison or not, etc.

3. **Mandatory System**: First the sentence for the given crime is stipulated and then it is decided whether the defendant is found guilty.

For the case of simplicity, consider three possible outcomes, death sentence, life sentence and release, and the following preferences of particular judges:

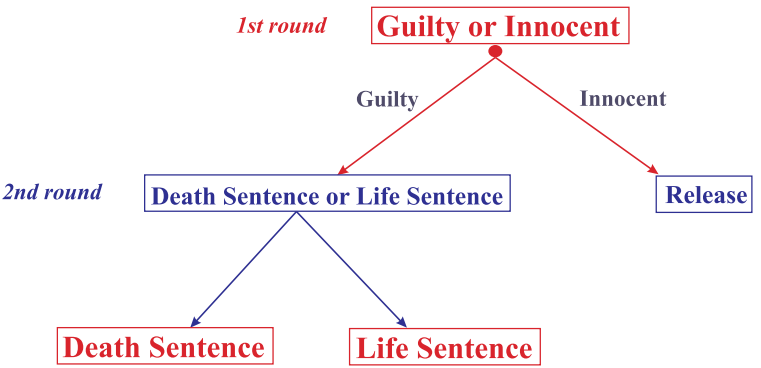
Ranking	Judge *A*	Judge *B*	Judge *C*
1.	Death Sentence	Life Sentence	Release
2.	Life Sentence	Release	Death Sentence
3.	Release	Death Sentence	Life Sentence

1. Status quo

In the first round the judges vote on defendant's guilt or innocence; if everybody just voted according to their rankings, "guilty" would win (judges A, B); in the second round, in the vote between life sentence and death sentence, death sentence would win (judges A, C). The first round is therefore in fact a vote between release and death sentence. Hence in the sophisticated voting, "release" therefore wins the first round (besides judge C, judge B will also vote for "release" in the first round since otherwise his less preferred outcome would occur).

1. Status Quo:

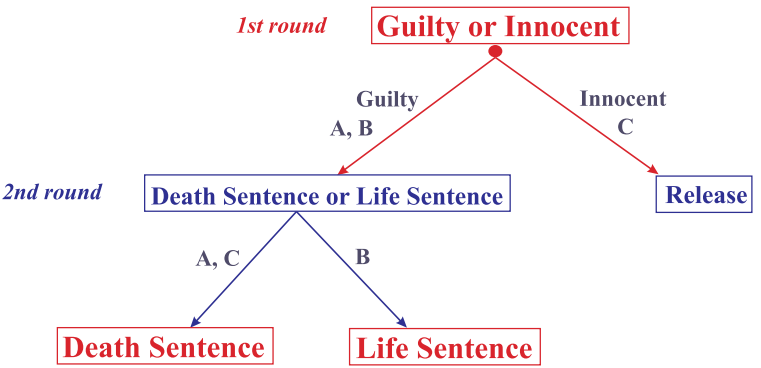
Judge A	Judge B	Judge C
+	#	♥
#	♥	+
♥	+	#



1. Status Quo:

Judge A	Judge B	Judge C
+	#	♥
#	♥	+
♥	+	#

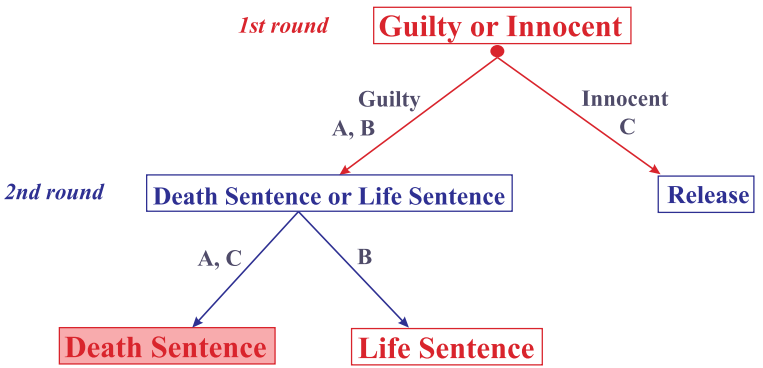
Sincere voting:



1. Status Quo:

Judge A	Judge B	Judge C
+	#	♥
#	♥	+
♥	+	#

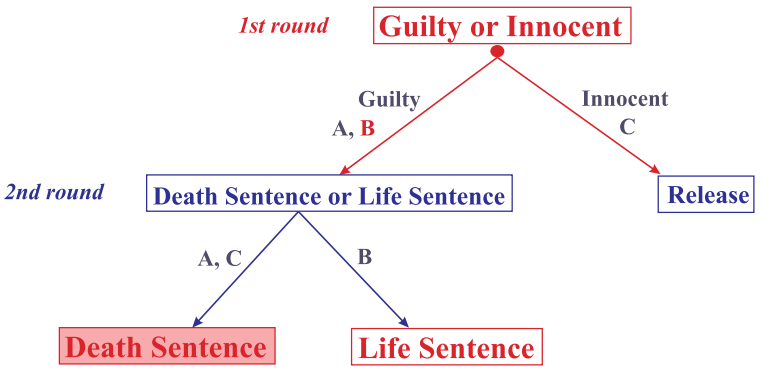
Sincere voting:



1. Status Quo:

Judge A	Judge B	Judge C
+	#	♥
#	♥	+
♥	+	#

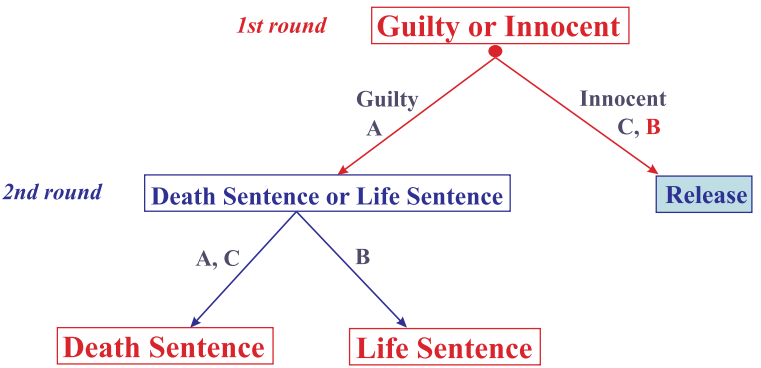
Sophisticated voting:



1. Status Quo:

Judge A	Judge B	Judge C
+	#	♥
#	♥	+
♥	+	#

Sophisticated voting:



2. Roman Tradition

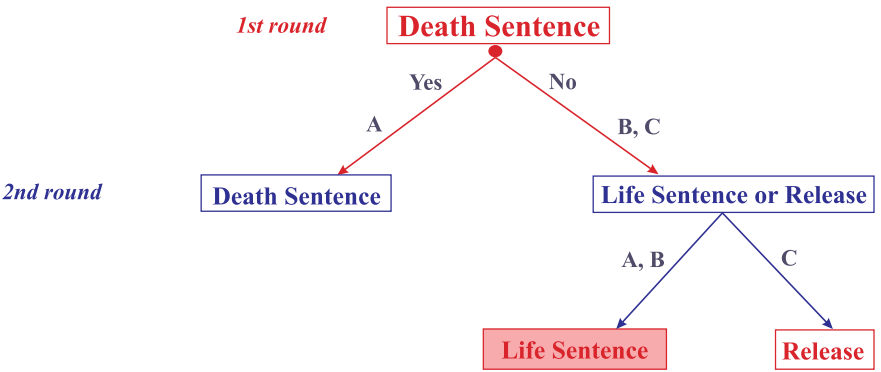
The first round is the vote on the most severe sentence, i.e. whether to impose the death sentence or not. If yes, the sentence is executed, if not, the second round occurs where the judges vote on life sentence or release.

Since in the second round life sentence would win (judges A, B), the first round is in fact a vote between death sentence and life sentence – in the sophisticated voting therefore **death sentence** wins (besides the judge A the judge C will vote for death sentence in the first round, since otherwise the second round would yield his less preferred outcome).

2. Roman Tradition:

Judge A	Judge B	Judge C
+	#	♥
#	♥	+
♥	+	#

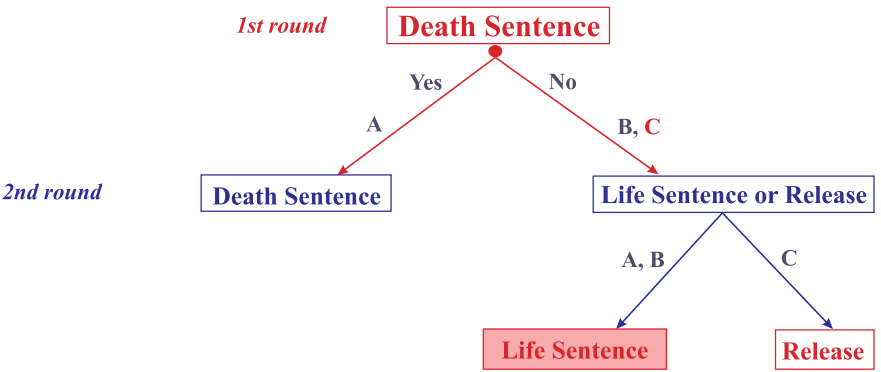
Sincere voting:



2. Roman Tradition:

Judge A	Judge B	Judge C
+	#	♥
#	♥	+
♥	+	#

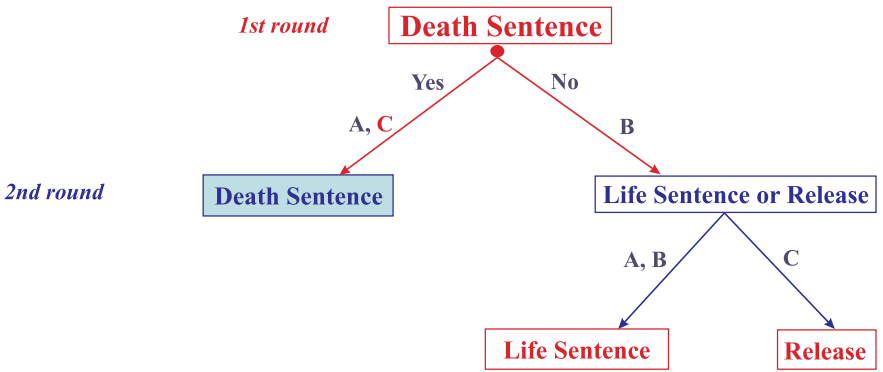
Sophisticated voting:



2. Roman Tradition:

Judge A	Judge B	Judge C
+	#	♥
#	♥	+
♥	+	#

Sophisticated voting:



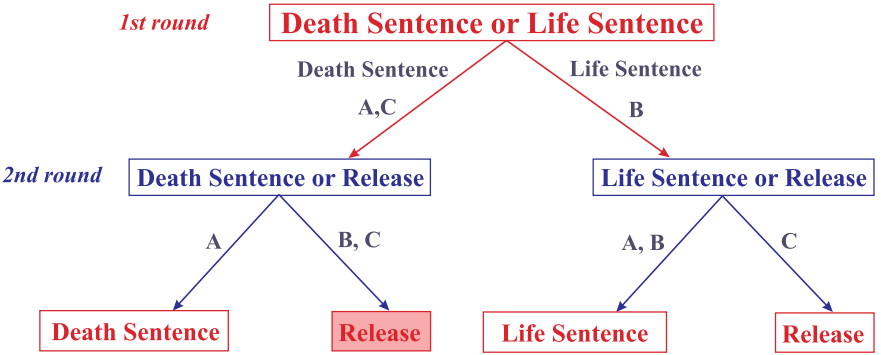
3. Mandatory System

The first round is a vote on the sentence for the given crime, in this case whether to impose death sentence or life sentence. The second round is a vote on whether to impose that sentence or not (release). In the decision between death sentence and release the second one would win (B, C), in the decision between life sentence and release life sentence would win (A, B). The first round is therefore a vote between release and life sentence, hence the defendant will be imposed the life sentence (A will vote for life sentence in the first round to avoid the less preferred outcome: release in the second round).

3. Mandatory System:

Judge A	Judge B	Judge C
+	#	♥
#	♥	+
♥	+	#

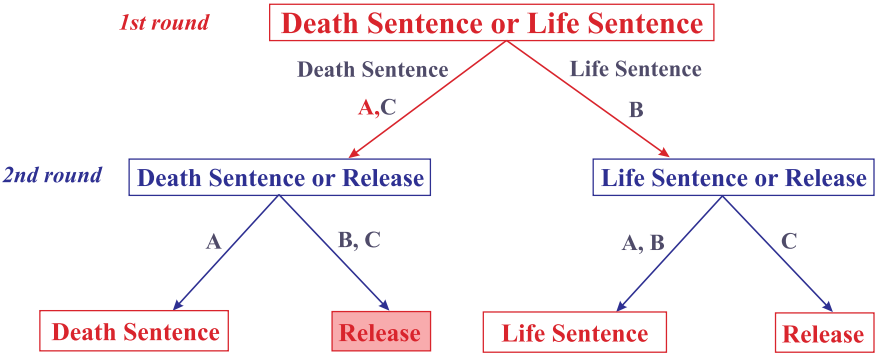
Sincere voting:



3. Mandatory System:

Judge A	Judge B	Judge C
+	#	♥
#	♥	+
♥	+	#

Sophisticated voting:



3. Mandatory System:

Judge A	Judge B	Judge C
+	#	♥
#	♥	+
♥	+	#

Sophisticated voting: