NORMAL FORM GAMES

☛ ***Example***. Advertising Strategies

Two mountain hotels in the same district, Krakonoš and Trautenberg, compete for tourists from three different countries: Germany, Czech Republic and Poland. The capacity of both hotels is sufficient for the accommodation of all tourists in only one of them. Both hotels have financial resources for the advertising campaign in only one country, the effectiveness of their campaigns is the same. If only one hotel runs the campaign in a given country, it gains all tourists from this country and hence the following profit: in the case of Germany 150 thousand EUR, in the case of the Czech Republic 90 thousand EUR and in the case of Poland 72 thousand EUR. If both firms run the campaign in the same country, receives each of them the half of the profit from this country's customers, similarly in the case that no firm runs the campaign in a specified country.

What are the optimal strategies for both hotels?

Germany	150
Czech Republic . . .	90
Poland	72
	312

Trautenberg

	Strategy	Germany	Czech Rep.	Poland
	Germany			
Krakonoš	Czech Rep.			
	Poland			

Germany	150
Czech Republic . . .	90
Poland	72
	312

Trautenberg

	Strategy	Germany	Czech Rep.	Poland
	Germany	$(\mathbf{156}, \mathbf{156})$		
Krakonoš	Czech Rep.		$(\mathbf{156}, \mathbf{156})$	
	Poland			$(\mathbf{156}, \mathbf{156})$

Germany	150
Czech Republic ...	90
Poland	72
	312

Trautenberg

	Strategy	Germany	Czech Rep.	Poland
	Germany	$(156, 156)$	$(186, 126)$	
Krakonoš	Czech Rep.		$(156, 156)$	
	Poland			$(156, 156)$

Germany	150
Czech Republic . . .	90
Poland	72
	312

Trautenberg

	Strategy	Germany	Czech Rep.	Poland
	Germany	$(\mathbf{156}, \mathbf{156})$	$(\mathbf{186}, \mathbf{126})$	
Krakonoš	Czech Rep.	$(\mathbf{126}, \mathbf{186})$	$(\mathbf{156}, \mathbf{156})$	
	Poland			$(\mathbf{156}, \mathbf{156})$

Germany	150
Czech Republic . . .	90
Poland	72
	312

	Strategy	Trautenberg: Germany	Czech Rep.	Poland
	Germany	$(156, 156)$	$(186, 126)$	$(195, 117)$
Krakonoš	Czech Rep.	$(126, 186)$	$(156, 156)$	
	Poland	$(117, 195)$		$(156, 156)$

Germany	150
Czech Republic ...	90
Poland	72
	312

Trautenberg

	Strategy	Germany	Czech Rep.	Poland
	Germany	(156, 156)	(186, 126)	(195, 117)
Krakonoš	Czech Rep.	(126, 186)	(156, 156)	(165, 147)
	Poland	(117, 195)	(147, 165)	(156, 156)

Germany	150
Czech Republic ...	90
Poland	72
	312

		Trautenberg		
	Strategy	Germany	Czech Rep.	Poland
	Germany	$\boxed{(156, 156)}$	$(186, 126)$	$(195, 117)$
Krakonoš	Czech Rep.	$(126, 186)$	$(156, 156)$	$(165, 147)$
	Poland	$(117, 195)$	$(147, 165)$	$(156, 156)$

Definition. n**-player normal form game (NFG)** is defined as the $(2n+1)$-tuple

$$(Q;\ S_1, \dots, S_n;\ u_1(s_1, \dots, s_n), \dots, u_n(s_1, \dots, s_n))\ ,$$

where $n \geq 2$ is a natural number; $Q = \{1, 2, \dots, n\}$ is a given finite set, so-called **set of players**, its elements are called **players**; for every $i \in \{1, 2, \dots, n\}$, S_i is an arbitrary set, so-called **set of strategies of the player** i, and

$$u_i : S_1 \times S_2 \times \cdots \times S_n \rightarrow \mathbb{R}$$

is a real function called **payoff function of the player** i.

☛ *Example*

Player 2

Player 1

Strategy	t_1		t_2
s_1	$\boxed{(2, 0)}$	$\leftarrow$	$(2, -1)$
	$\uparrow$		$\downarrow$
s_2	$(1, 1)$	$\leftarrow$	$(3, -2)$

Example

Player 2

Player 1

Strategy	t_1		t_2
s_1	$\boxed{(2, 0)}$	$\leftarrow$	$(2, -1)$
	$\uparrow$		$\downarrow$
s_2	$(1, 1)$	$\leftarrow$	$(3, -2)$

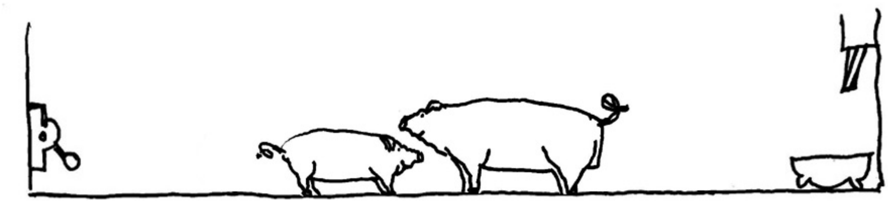
Definition. An n-tuple of strategies **s**$^* = (s_1^*, \ldots, s_n^*)$ is called an **equilibrium point** or **Nash equilibrium** of the game (HNT), if and only if for every $i \in \{1, 2, \ldots, n\}$ end every $s_i \in S_i$ the following condition holds:

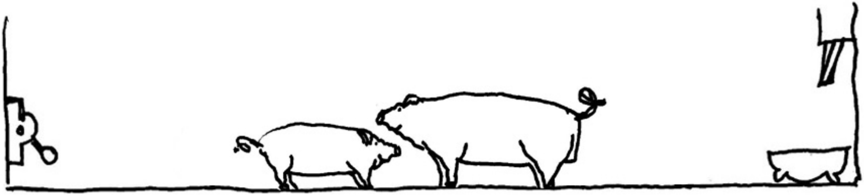
$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \leq$$

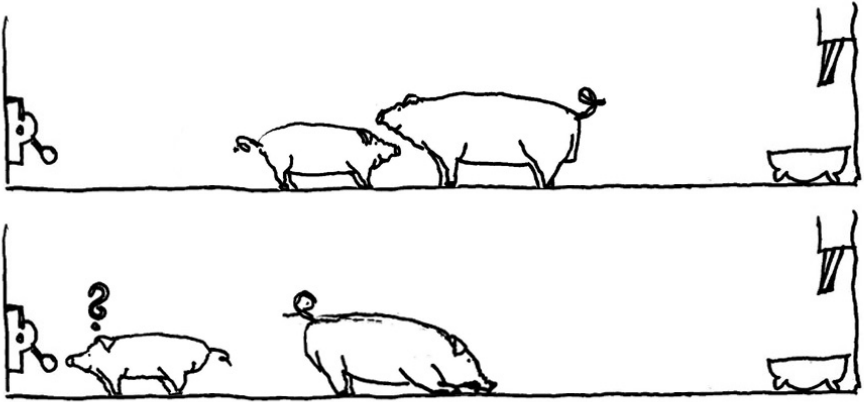
$$\leq u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*). \quad (1.1)$$

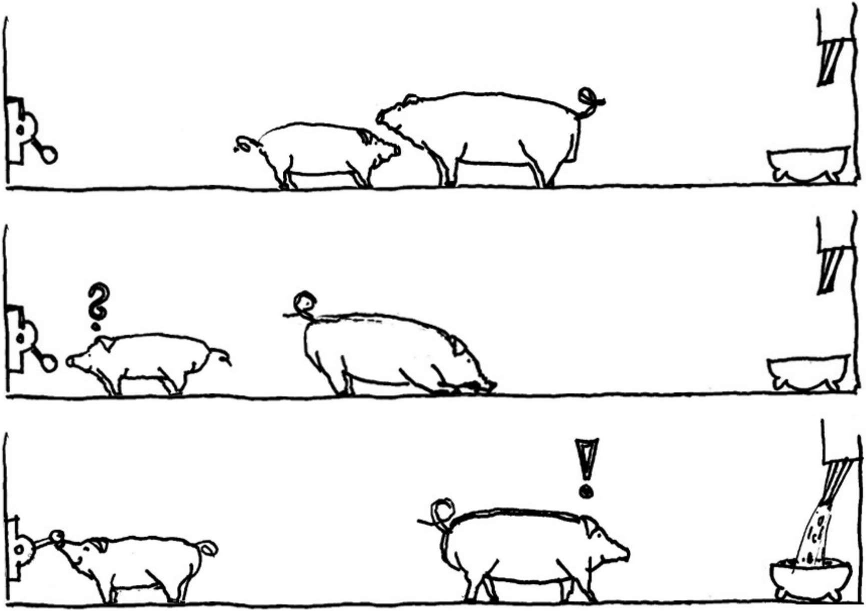
1979 B. A. Baldwin, G. B. Meese: Skinner sty

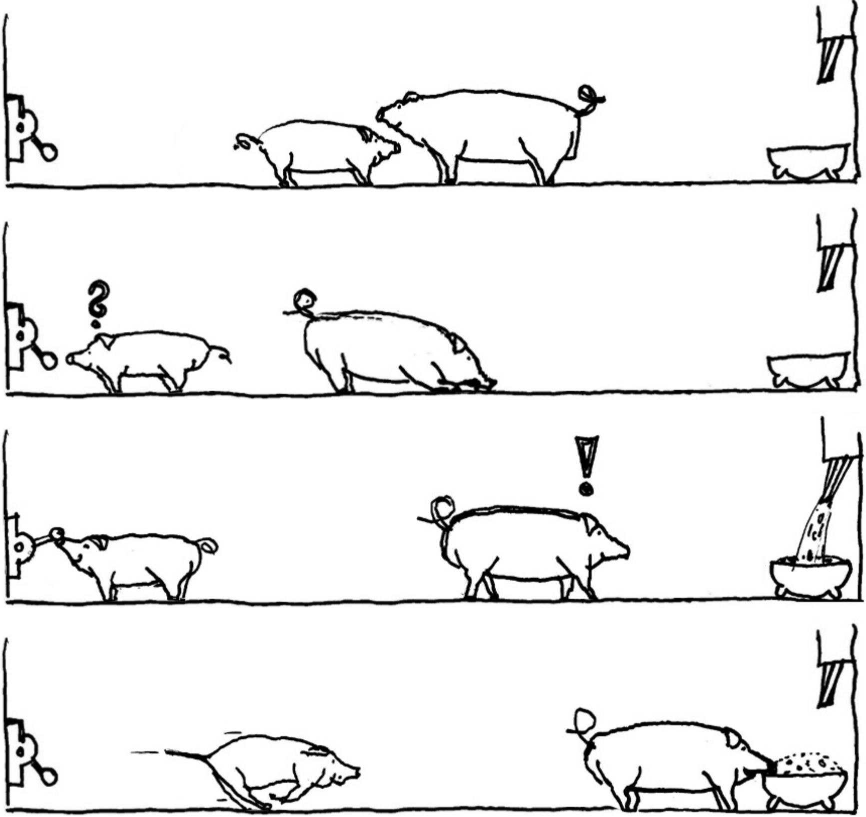
1979 B. A. Baldwin, G. B. Meese: Skinner sty

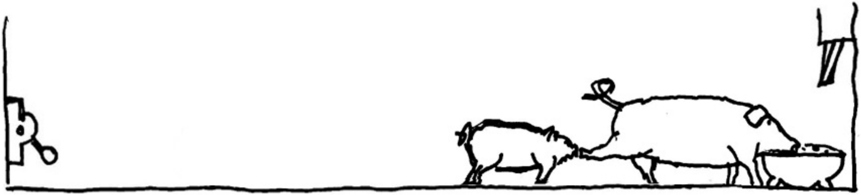


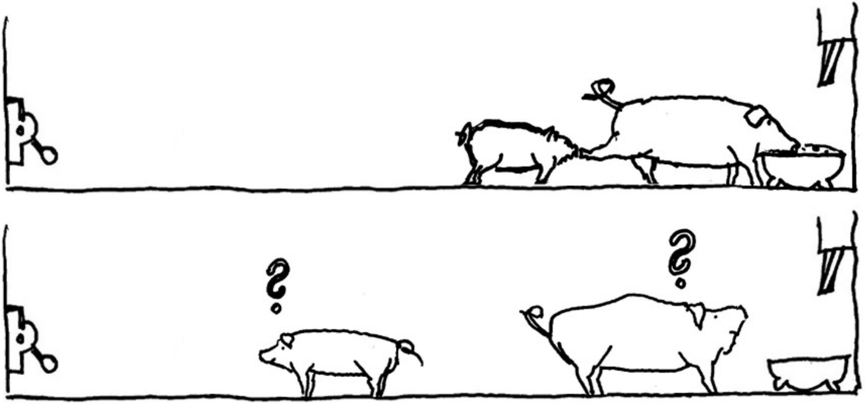


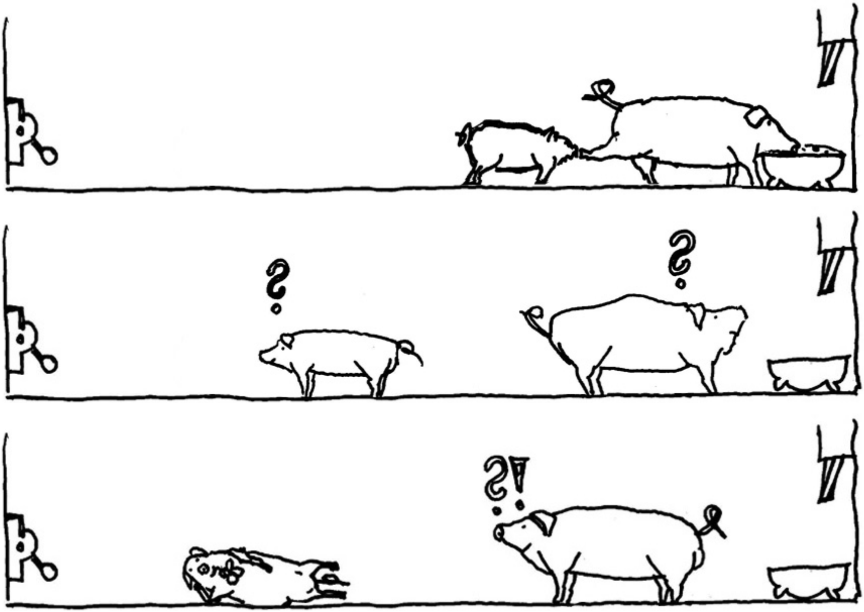


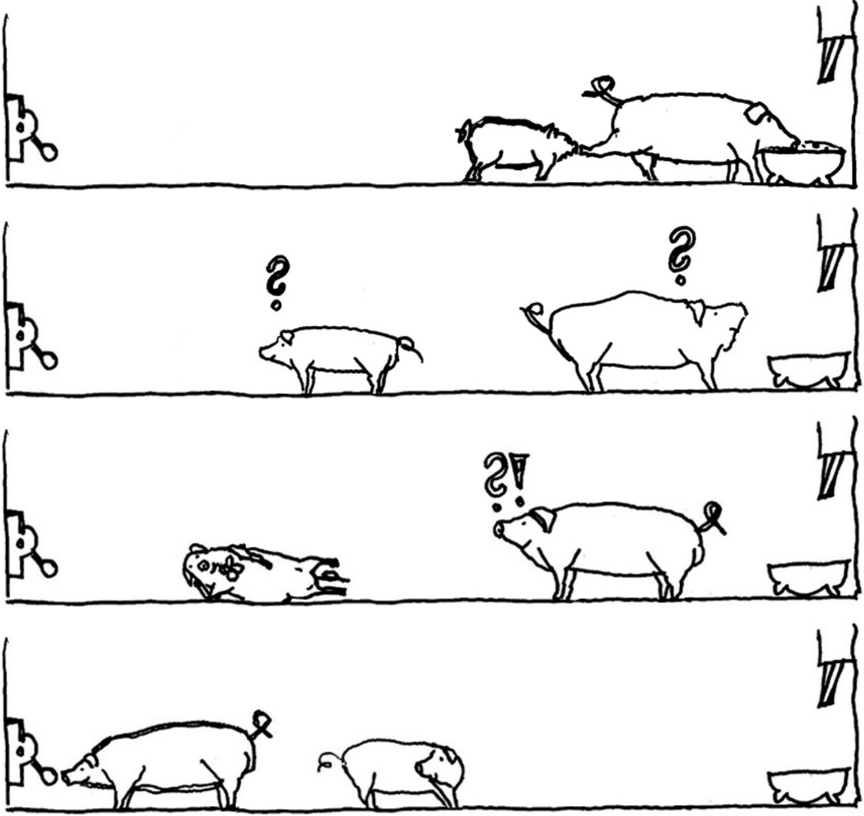


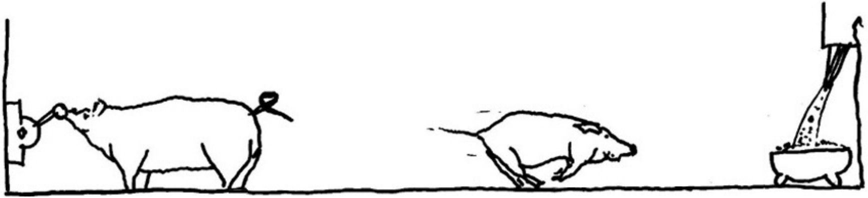


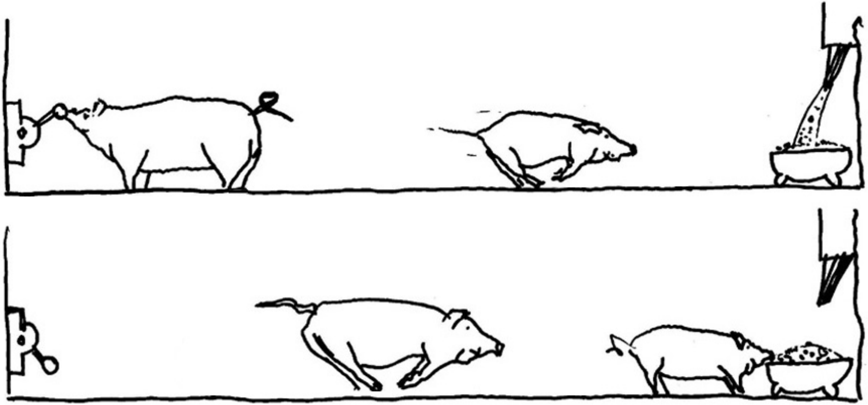


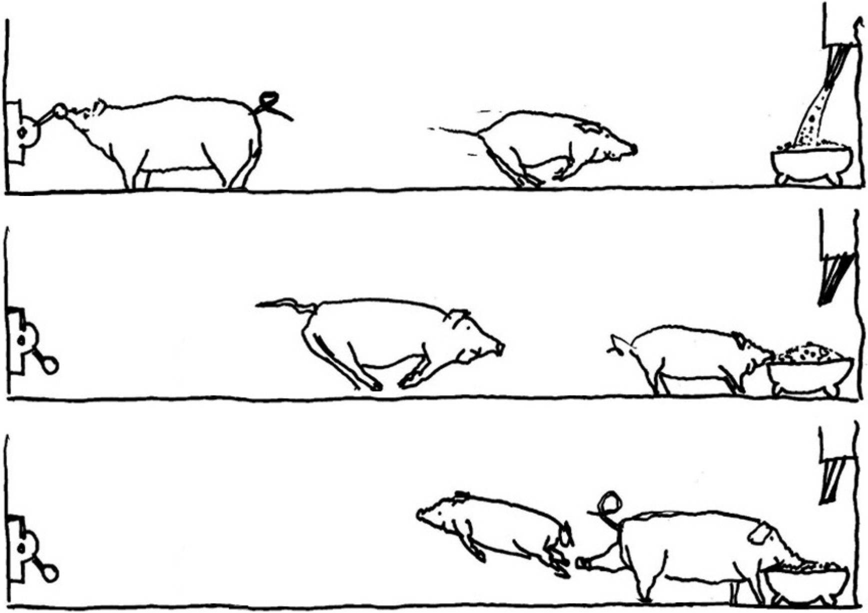


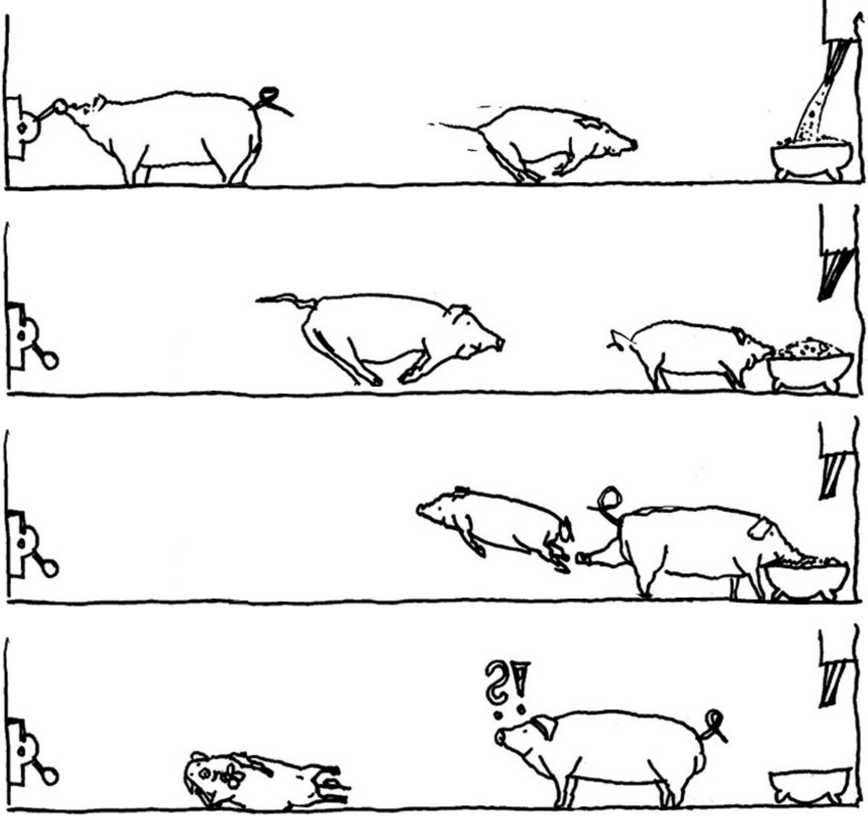


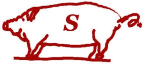


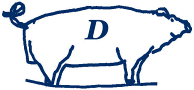












Strategy	Press the lever		Sit by the trough
Press the lever	$(8, -2)$	$\longrightarrow$	$\boxed{(5, 3)}$
Sit by the trough	$(10, -2)$	$\longrightarrow$	$(0, 0)$

☛ *Example*

Player 2

Player 1

$$\begin{array}{c|ccc}
\text{Strategy} & t_1 & & t_2 \\
\hline
s_1 & (1,-1) & \longrightarrow & (-1,1) \\
 & \uparrow & & \downarrow \\
s_2 & (-1,1) & \longleftarrow & (1,-1)
\end{array}$$

☛ *Example*

Player 2

Strategy	t_1		t_2	
s_1	$(1, -1)$	$\longrightarrow$	$(-1, 1)$	p
	$\uparrow$		$\downarrow$	
s_2	$(-1, 1)$	$\longleftarrow$	$(1, -1)$	$1 - p$
	q		$1 - q$	

Player 1

☛ ***Example***

Strategy	t_1		t_2	
s_1	$(1,-1)$	$\longrightarrow$	$(-1,1)$	p
	$\uparrow$		$\downarrow$	
s_2	$(-1,1)$	$\longleftarrow$	$(1,-1)$	$1-p$
	q		$1-q$	

Player 2 (columns), Player 1 (rows)

Mixed strategies:

$$\{s_1, s_2\} \rightsquigarrow \{(p, 1-p), p \in \langle 0, 1 \rangle\}$$

$$\{t_1, t_2\} \rightsquigarrow \{(q, 1-q), q \in \langle 0, 1 \rangle\}$$

Payoff functions u_1, u_2 $\rightsquigarrow$ expected payoffs π_1, π_2 :

$$\pi_1(p,q) = -1pq + 1(1-p)q + 1p(1-q) - 1(1-p)(1-q)$$

$$\pi_2(p,q) = \quad 1pq - 1(1-p)q - 1p(1-q) + 1(1-p)(1-q)$$

Finite Normal Form Game

Finite normal form game is a normal form game where all the sets $S_1, S_2, \ldots, S_n$ are finite.

Definition. Consider a finite n-player normal form game. Denote the number of elements of the strategy set S_i of an arbitrary player i by the symbol m_i. **Mixed strategy** of the player i is defined as the vector of probabilities

$$p^i = (p^i_1, p^i_2, \ldots, p^i_{m_i}), \qquad \text{with} \quad p^i_j \geq 0 \quad \text{for all} \quad 1 \leq j \leq m_i,$$

$$\sum_{j=1}^{m_i} p^i_j = 1, \tag{1.2}$$

where p^i_j, $j = 1, 2, \ldots, m_i$, expresses the probability of choosing the j-th strategy from the strategy space S_i.

A mixed strategy is therefore again a certain strategy that can be characterized in the following way:

"Use the strategy $s_1^i \in S_i$ *with the probability* p_1^i,

. . . ,

use the strategy $s_{m_i}^i \in S_i$ *with the probability* $p_{m_i}^i$*."*

For the case of distinction, the elements of the strategy set S_i are called **pure strategies**.

Theorem (J. Nash). *In mixed strategies, every finite normal form game has at least one equilibrium point.*

MATHEMATIZATION OF GAME THEORY

JOHN VON NEUMANN (1903 – 1957)

1926 proof of **minimax theorem** (Gött.Math.Soc.)

1928 **Sur la théorie des jeux** (Comptes Rendus)
Zur Theorie der Gesellschaftsspiele (Math. Annalen)

- **Mathematization of strategy games**
- **The proof of ”minimax theorem”**

Formulation: **finite n-player zero sum game**

More results: $n = 2$

$$(\{1,2\}; \{s_1, \ldots, s_k\}, \{t_1, \ldots, t_l\}; u_1, u_2)$$

$$u_1(s_i, t_j) + u_2(s_i, t_j) = 0$$

$$
\begin{array}{cc}
 & \textbf{Player 2} \\
 & \begin{array}{cccc} t_1 & t_2 & \dots & t_l \end{array} \\
\textbf{Player 1} \quad \begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_k \end{array} & \begin{pmatrix} u_1(s_1,t_1) & u_1(s_1,t_2) & \dots & u_1(s_1,t_l) \\ u_1(s_2,t_1) & u_1(s_2,t_2) & \dots & u_1(s_2,t_l) \\ \dots & \dots & \dots & \dots \\ u_1(s_k,t_1) & u_1(s_k,t_2) & \dots & u_1(s_k,t_l) \end{pmatrix}
\end{array}
$$

Player 1: $\min_{t_j} u_1(s_i, t_j) \rightsquigarrow$ **MAX**

Player 2: $\max_{s_i} u_1(s_i, t_j) \rightsquigarrow$ **MIN**

It is:

$$\max_{s_i} \min_{t_j} u_1(s_i, t_j) \leq \min_{t_j} \max_{s_i} u_1(s_i, t_j)$$

Player 2

$$
\textbf{Player 1} \quad
\begin{array}{c c c}
 & \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} & \\
\begin{array}{c} s_1 \\ s_2 \\ s_k \end{array} &
\begin{pmatrix} 5 & 4 & \boxed{4} & 5 \\ -4 & 5 & 3 & 9 \\ 7 & 8 & -1 & 8 \end{pmatrix} &
\begin{array}{c} \boxed{4} \\ -4 \\ -1 \end{array} \quad \text{min} \\
\text{max:} & \begin{array}{cccc} 7 & 8 & \boxed{4} & 9 \end{array} &
\end{array}
$$

$$\max_{s} \min_{t} u_1(s_i, t_j) = 4 = \min_{t} \max_{s} u_1(s_i, t_j)$$

Player 2

Player 1

$$
\begin{array}{cc}
 & \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\
\begin{array}{c} s_1 \\ s_2 \\ s_k \end{array} & \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \\
\text{max:} & \begin{array}{ccc} 1 & 1 & 1 \end{array}
\end{array}
\begin{array}{c} -1 \\ -1 \\ -1 \end{array} \quad \text{min}
$$

$$\max_s \min_t u_1(s_i, t_j) = -1 < \min_t \max_s u_1(s_i, t_j) = 1$$

⇝ **Mixed strategies** – expected payoff for player 1:

$$\pi_1(p, q) = \sum_{i=1}^{k} \sum_{j=1}^{l} u_1(s_i, t_j) p_i q_j$$

Theorem. There always exist mixed strategies (p^*, q^*), such that

$$\pi_1(p^*, q^*) = \max_p \min_q \pi_1(s_i, t_j) = \min_q \max_p \pi_1(s_i, t_j)$$

GAME THEORY = MATHEMATICAL DISCIPLINE

John von Neumann (1903 – 1957), **Oskar Morgenstern (1902 – 1976)**

1944 Theory of Games and Economic Behavior

- **Application possibilities of game theory** – detailed formulation of economical problem
- **Axiomatic utility theory**
- **General formal description of a game of strat.**
- **2-player antagonistic finite games**
- **n-player cooperative games** (with transferable payoffs)

  $\rightsquigarrow$ **von Neumann-Morgenstern's solution**

  (it is not unique, does not necessarily exist)

  $\dots$

$\rightsquigarrow$ **Massive development of game theory and its applications**

The next step: Non-constant sum noncooperative games, cooperative games without transferable payoff

$$\begin{pmatrix} (3,-3) & \boxed{(2,-2)} \\ (0,0) & (1,-1) \end{pmatrix}$$

$$\begin{pmatrix} (3,3) & \boxed{(2,4)} \\ (0,6) & (1,5) \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} (3,3) & \longrightarrow & \boxed{(2,4)} \\ \uparrow & & \downarrow \\ (0,2) & \longrightarrow & \underline{\underline{(4,5)}} \end{pmatrix}$$

$(4,5)$ . . . **mutually best replies – equilibrium point**

JOHN FORBES NASH (*1928)

Equilibrium Concept

1949 ***Non-Cooperative Games*** (thesis; Ph.D. 1950)

- **Nash equilibrium** – introduction of the concept
- **Nash theorem** – proof of its existence

  Motivations:

  - Rational considerations of payoffs
  - Interactive adjustment processes, in which boudedly rational agents observe the strategies played by their likely opponents over time, and gradually learn to adjust their own strategies to earn higher payoffs, will eventually converge to a Nash equilibrium – if their converge to anything at all (only in the thesis)

  $\leadsto$ **confirmed by later experimental works.**

1950 ***Equilibrium Points in n-Person Games***
Proceedings of Nat. Acad. of Sciences of USA

1951 ***Non-Cooperative Games,*** Annals of Math.

Axiomatic Bargaining Theory

1950 ***The Bargaining Problem,*** Econometrica 18

1953 ***Two-Person Cooperative Games,*** ibid. 21

- Non-transferable payoffs
- The approach to cooperative games by their reduction to non-cooperative
- Axioms that a solution shall satisfy
- The proof of the existence of the unique solution: **Nash bargaining solution**

BEGINNINGS OF UTILITY THEORY

Daniel Bernoulli (1700 – 1782)

Exposition of a New Theory of Risk Evaluation

(Petersburg 1725 – 1733, publ. 1838)

A gamble should be evaluated not in terms of the value of its alternative pay-offs but rather in terms of the value of its **utilities** .

Illustration example: *Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing ofr twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats? Would he not be ill-advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is negative. On the other hand I am inclined to believe that a rich man would be ill-advised to refuse to buy the lottery ticket for nine thousand ducats. If I am not wrong then it seems clear that all men cannot use the same rule to evaluate the gamble . . .*

Utility function $u(x)$

... amount of utility units the number of *utility units* for the financial amount x;

Assumption:

the increase of utility $du(x)$ is proportionate to the increase of the amount dx and inversely proportionate to the quantity previously possessed (a poor man generally obtains more utility than does a rich man from an equal gain):

$$
\begin{aligned}
du(x) &= \frac{b\,dx}{x} && b > 0 \quad \text{(constant of proportion)} \\
u(x) &= b \ln x + c && c \in \mathbb{R} \\
&= b \ln x - b \ln \alpha && \alpha \in (0, +\infty) \\
u(x) &= b \ln \frac{x}{\alpha} && \alpha \text{ – quantity originally possessed}
\end{aligned}
$$

Application: elucidation of **St. Petersburg Paradox:**

St. Petersburg Paradox

Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. Suppose we seek to determine the value of Paul's expectation.

The mean value of the win:

$$\frac{1}{2}+2\cdot\left(\frac{1}{2}\right)^{2}+\cdots+2^{n-1}\cdot\left(\frac{1}{2}\right)^{n}+\cdots=\frac{1}{2}+\frac{1}{2}+\cdots\frac{1}{2}+\cdots=\infty$$

Paradox:

although an expected value of the win is infinite, a reasonable person sells – with a great pleasure – the engagement in the play for 20 ducats.

Bernoulli: the mean value of the utility brought by the win:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} b \ln \frac{\alpha + 2^{n-1}}{\alpha} =$$

$$= b \ln[(\alpha+1)^{\frac{1}{2}} (\alpha+2)^{\frac{1}{4}} \cdots (\alpha + 2^{n-1})^{\frac{1}{2^n}} \cdots] - b \ln \alpha$$

Amount D, whose addition to the initial possession brings the same utility:

$$b \ln \frac{\alpha + D}{\alpha} = b \ln[(\alpha+1)^{\frac{1}{2}} (\alpha+2)^{\frac{1}{4}} \cdots (\alpha+2^{n-1})^{\frac{1}{2^n}} \cdots] - b \ln \alpha$$

$$D = [(\alpha+1)^{\frac{1}{2}} (\alpha+2)^{\frac{1}{4}} \cdots (\alpha + 2^{n-1})^{\frac{1}{2^n}} \cdots] - \alpha$$

For a zero initial possession:

$$D = \sqrt[2]{1} \cdot \sqrt[4]{2} \cdot \sqrt[8]{4} \cdot \sqrt[16]{8} \cdots = 2$$

Imperfections of Bernoulli's utility function

- Defined for positive values of q only, while in the real world the losses are important, too
- Utility function is different for different people, depends on other than property conditions, too

Important stimulus for further development.

Similar – but independent – considerations
(Bernoulli cites at the end of his treatise):

Gabriel Cramer (1704 – 1752)

Letter to Nicholas Bernoulli, 1728

Idea: people evaluate money in proportion to the utility they can obtain from it.

Assumption:

Any amount above 10 millions, or (for the sake of simplicity) above 2^{24} *ducats be deemed by me equal in value to* 2^{24} *ducats or, better yet, that I can never win more than that amount, no matter how long it takes before the coin falls with its cross upward* [without any further discussion]. *In this case, my expectation is*

$$\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 2+\frac{1}{8}\cdot 4+\cdots+\frac{1}{2^{24}}\cdot 2^{24}+\frac{1}{2^{25}}\cdot 2^{24}+\frac{1}{2^{26}}\cdot 2^{24}+\cdots=$$

$$=\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=12+1=13\,.$$

Thus, my moral expectation is reduced in value to 13 ducats and the equivalent to be paid for it is similarly reduced – a result which seems much more reasonable than does rendering it infinite.

Daniel Bernoulli
(1700 – 1782)

Gabriel Cramer
(1704 – 1752)

Normal Form Games whose Strategy Sets are Open Intervals

Theorem – EQUILIBRIUM TEST.
Let G be a normal form game whose strategy sets are open intervals and payoff functions are twice differentiable. Assume that a strategy profile $(s_1^*, \dots, s_n^*)$ satisfies the following conditions:

1) $$\frac{\partial u_i(s_1^*, \dots, s_n^*)}{\partial s_i} = 0 \quad \text{for each } i = 1, 2, \dots, n.$$

2) *Each s_i^* is the only stationary point of the function*

$$u_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*), \quad s_i \in S_i.$$

3) $$\frac{\partial^2 u_i(s_1^*, \dots, s_n^*)}{\partial s_i^2} < 0 \quad \text{for each } i = 1, 2, \dots, n.$$

Then $(s_1^*, \dots, s_n^*)$ is an equilibrium point of the game G.

Remark. In practice, we usually find the solution of the system of equations

$$\frac{\partial u_i(s_1^*, \ldots, s_n^*)}{\partial s_i} = 0 \qquad \text{for each } i = 1, 2, \ldots, n,$$

and then use other economic considerations to verify that the solution is the Nash equilibrium of the game.

A typical example of the normal form game whose strategy sets are open intervals is an oligopoly model where several firms produce the same product, each of them contributing significantly to the total production. The price is given by the demand equation which describes the market behavior and gives the relation between the price and the total number of products that should be sold. In other words, it gives the highest price for which it is possible to sell the given amount of products. In Cournot models discussed below the demand equation has the simplest possible form:

$$\boldsymbol{p + q = M, \qquad M >> c,} \tag{1.3}$$

where $\boldsymbol{p}$ is the price of the product, $\boldsymbol{q}$ is the demand for this product in the market, $\boldsymbol{c}$ expresses the production costs of one piece, $\boldsymbol{M}$ is a constant much greater than $\boldsymbol{c}$.

☛ *Example*. Cournot Monopoly Model

If only one firm – a monopolist – produces the given product, the situation is clear. The monopolist knows that when he produces q products, the highest price for which he can sell one piece to sell all the production out is given by the demand equation (1.3):

$$p = M - q. \tag{1.4}$$

Since nobody else influences the total amount of products, the monopolist is faced with the problem of a mere profit maximization, i.e. finding the maximum of one variable function

$$u(q) = p \cdot q - c \cdot q = Mq - q^2 - cq = (M - q - c)q. \tag{1.5}$$

Maximum can easily be found with the help of a derivative:

$$\begin{aligned} u'(q) = M - c - 2q = &\ 0 \\ q^*_{mon} = &\ \underline{\underline{\tfrac{1}{2}(M - c)}} \end{aligned} \tag{1.6}$$

Since for $q < q^*_{mon}$ the derivative is positive and the function u increasing, for $q > q^*_{mon}$ is the derivative negative and the function u decreasing, it is really a maximum. Hence the monopolist reaches the maximal profit by the production of $q^*_{mon} = \frac{1}{2}(M - c)$ pieces, namely

$$u^*_{mon} = u(q^*_{mon}) = \left(M - \tfrac{1}{2}(M-c) - c\right)\tfrac{1}{2}(M-c) = \\ = \underline{\underline{\left[\tfrac{1}{2}(M-c)\right]^2}} \tag{1.7}$$

The corresponding price is then

$$p^*_{mon} = \underline{\underline{\tfrac{1}{2}(M+c)}} \tag{1.8}$$

☛ *Example*. Cournot Duopoly Model (1938).

Now consider two firms producing the same product, each of them contributing significantly to the total production. Monopolist's problem lied in finding the maximum of a simple quadratic function Duopolists are faced with a game, since each of them controls only a part of the total production; the price he receives for his products therefore depends not only on his own decision, but on the decision of the opponent, too. Now we assume that the duopolists decide simultaneously and independently of each other.

Let us find an equilibrium point in this game, i.e. an optimal amounts that should be produced by particular duopolists, when it is not advantageous for none of them to deviate.

Denote q_1, q_2 the production of the first and the second duopolist respectively. Maximal price for which all the products can be sold is again given by the demand equation (1.3):

$$p = M - q_1 - q_2 \tag{1.9}$$

The situation can be modelled by the normal form game where **players** are duopolists, each of them choosing in general the number of an interval $(0, M)$; **strategy sets** are therefore the intervals

$$S_1 = S_2 = (0, M),$$

payoff functions are the profits:

$$\begin{aligned} u_1(q_1, q_2) &= (p - c)q_1 = (M - c - q_1 - q_2)q_1 \\ u_2(q_1, q_2) &= (p - c)q_2 = (M - c - q_1 - q_2)q_2 \end{aligned} \tag{1.10}$$

Although the strategy sets are infinite, the equilibrium point can be found very simply. The first duopolist looks for a function which to each opponent's strategy, i.e. each number q_2 assigns such a number $q_1 = R_1(q_2)$, which is the best reply to q_2 in the sence that the value of the function $u_1(q_1, q_2)$ is maximal. In other words, for every fixed (but arbitrary) $q_2 \in S_2$ the first duopolist searches the maximum of the function $u_1(q_1, q_2)$ which is now a function of one variable q_1 :

$$\frac{\partial u_1}{\partial q_1} = M - c - q_2 - 2q_1 = 0$$

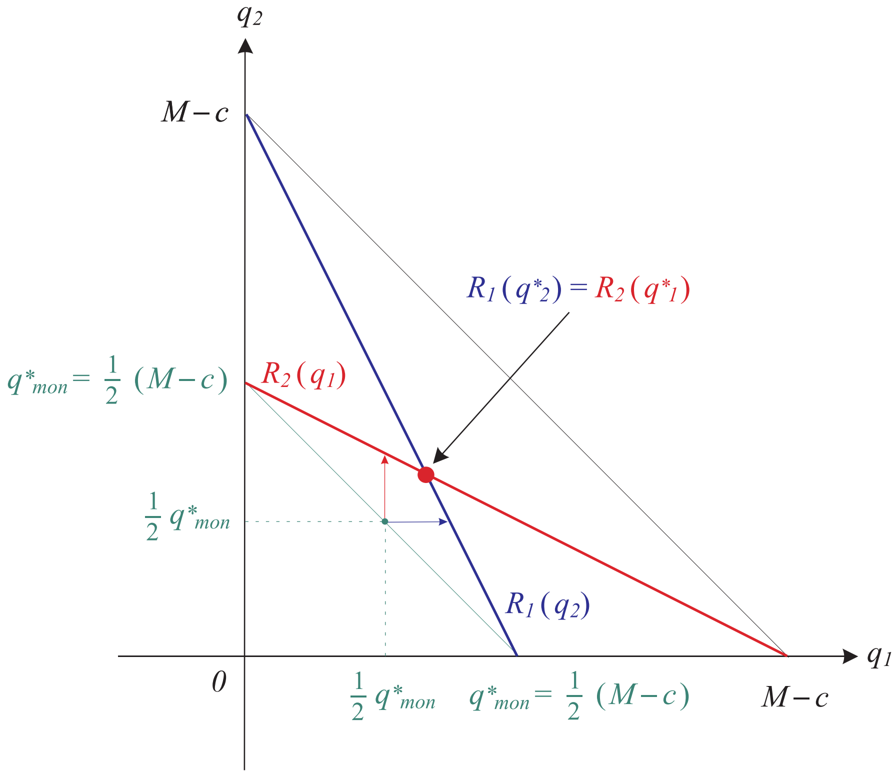
$$R_1(q_2) = q_1 = \tfrac{1}{2}(M - c - q_2) \qquad (1.11)$$

(verify this is really a maximum). Similarly, for every strategy q_1 of the first duopolist, the second duopolist searches his best reply $q_2 = R_2(q_1)$, i.e. such number which for a given q_1 maximizes the profit u_2 :

$$\frac{\partial u_2}{\partial q_2} = M - c - q_1 - 2q_2 = 0$$

$$R_2(q_1) = q_2 = \tfrac{1}{2}(M - c - q_1) \qquad (1.12)$$

Functions $R_1(q_2)$ and $R_2(q_1)$ are called **reaction curves**; we can represent them in the following way:



q_2
$M-c$
$R_1(q^*_2) = R_2(q^*_1)$
$q^*_{mon} = \frac{1}{2}(M-c)$
$R_2(q_1)$
$\frac{1}{2}q^*_{mon}$
$R_1(q_2)$
q_1
0
$\frac{1}{2}q^*_{mon}$
$q^*_{mon} = \frac{1}{2}(M-c)$
$M-c$

From the definition it can be easily deduced that (q_1^*, q_2^*) is an equilibrium point if and only if it is an intersection of reaction curves:

$$q_1^* = R_1(q_2^*) \qquad \text{end} \qquad q_2^* = R_2(q_1^*).$$

In our example:

$$(q_1^*, q_2^*) = \underline{\underline{\left(\tfrac{1}{3}(M-c), \tfrac{1}{3}(M-c)\right)}}. \tag{1.13}$$

The price for which duopolists will sell the products is then

$$p = M - \tfrac{2}{3}(M-c) = \underline{\underline{\tfrac{1}{3}M + \tfrac{2}{3}c}}. \tag{1.14}$$

The corresponding profit for each duopolist is

$$u_1(q_1^*, q_2^*) = u_2(q_1^*, q_2^*) = \underline{\underline{\left[\tfrac{1}{3}(M-c)\right]^2}}. \tag{1.15}$$

In equilibrium the total production of duopolists is

$$q_D^* = q_1^* + q_2^* = \tfrac{2}{3}(M-c) > \tfrac{1}{2}(M-c) = q_{mon}^* \ ,$$

hence they sell the greater total production for the lower price than a monopolist.

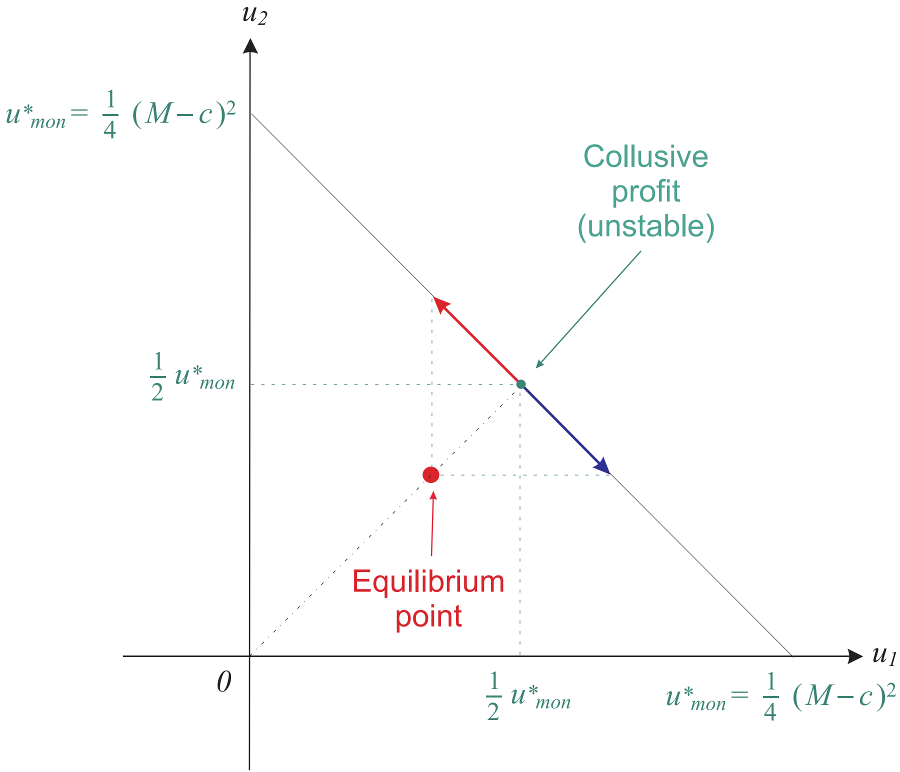
Comparing the results for the monopoly and duopoly, it is clear that it would be the best for duopolists to get together in a cosy cartel and collude to avoid the inroads into their profits that competition brings. Then they could together produce only

$$q_1 + q_2 = q^*_{mon} = \tfrac{1}{2}(M - c) \tag{1.16}$$

(these points form the green line segment) and with respect to circumstances divide the profit that emerged in this way – in symmetrical situation fifty-fifty:

$$\left(\tfrac{1}{2}q^*_{mon}, \tfrac{1}{2}q^*_{mon}\right) = \left(\tfrac{1}{4}(M - c), \tfrac{1}{4}(M - c)\right).$$

Nevertheless, this outcome is **unstable**, because for each duopolist it is profitable to deviate to his best reply to the opponent's choice and receive more for himself.



u_2
$u^*_{mon} = \frac{1}{4}\,(M - c)^2$
Collusive profit (unstable)
$\frac{1}{2}\,u^*_{mon}$
Equilibrium point
u_1
0
$\frac{1}{2}\,u^*_{mon}$
$u^*_{mon} = \frac{1}{4}\,(M - c)^2$

The problem lies in the fact that such deals are collusive and, with respect to anti-monopoly laws, usually lawless – and a collusive deal made in smoke-filled hotel room is cheap and certainly not binding.

In the end – fortunately for the customer (and for this reason there are those anti-monopoly laws) – the only deal when no duopolist has a temptation to deviate, is the above mentioned **equilibrium point**

$$(q_1^*, q_2^*) = \left(\tfrac{2}{3} q_{mon}^*, \tfrac{2}{3} q_{mon}^*\right) = \left(\tfrac{1}{3}(M-c), \tfrac{1}{3}(M-c)\right).$$

But the situation changes radically when the game is **repeated**, i.e. when the same duopolists find themselves in the same situation repeatedly: if the probability that there will be one more round is high in each time the game is played, it can be advantageous for each duopolist to keep the deal.

☛ *Example*. Cournot Oligopoly Model

Consider n firms producing the same product, each of them contributing significantly to the total production. Now we have an n-player game where each player searches the optimal amount of products q_i he shall produce. Let us find the equilibrium point of this game.

Likewise in the case of duopoly, profits of particular oligopolists are the following:

$$
\begin{aligned}
u_1(q_1, \dots, q_n) &= (p - c)\, q_1 = (M - c - q_1 - q_2 - \cdots - q_n)\, q_1 \\
u_2(q_1, \dots, q_n) &= (p - c)\, q_2 = (M - c - q_1 - q_2 - \cdots - q_n)\, q_2 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
u_n(q_1, \dots, q_n) &= (p - c)\, q_n = (M - c - q_1 - q_2 - \cdots - q_n)\, q_n
\end{aligned}
\tag{1.17}
$$

Again, it is easy to find the equilibrium point. From the conditions

$$\begin{aligned}
\frac{\partial u_1}{\partial q_1} &= M - c - 2q_1 - q_2 - \cdots - q_n = 0 \\
\frac{\partial u_2}{\partial q_2} &= M - c - q_1 - 2q_2 - \cdots - q_n = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial u_n}{\partial q_n} &= M - c - q_1 - q_2 - \cdots - 2q_n = 0
\end{aligned}$$

we get the system of equations:

$$\begin{array}{ccccccccc}
2q_1 & + & q_2 & + & \cdots & + & q_n & = & M - c \\
q_1 & + & 2q_2 & + & \cdots & + & q_n & = & M - c \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
q_1 & + & q_2 & + & \cdots & + & 2q_n & = & M - c
\end{array}$$

Its solution is:

$$q_1^* = q_2^* = \cdots = q_n^* = \underline{\underline{\frac{M-c}{n+1}}} \tag{1.18}$$

Oligopolists therefore produce the total production

$$q_O^* = q_1^* + q_2^* + \cdots + q_n^* = n\frac{M-c}{n+1} = \underline{\underline{\frac{n}{n+1}(M-c)}} \tag{1.19}$$

From the result it is clear that the greater the number of firms is, the greater is the total production q_O^* and the lower is the price p^* and the total profit u^* of the firms:

$$p^* = \underline{\underline{\frac{1}{n+1}M + \frac{n}{n+1}c}} \tag{1.20}$$

$$u^* = \underline{\underline{\frac{n}{(n+1)^2}(M-c)^2}} \tag{1.21}$$

A limit case of oligopoly where $n \to \infty$ is a **perfect competition**: here the great number of small firms contribute to the total production and no firm can significantly influence the total outcome.

The total production is now given by

$$\lim_{n\to\infty} \frac{n}{n+1}(M-c) = \underline{\underline{(M-c)}}, \tag{1.22}$$

the price of a product in the market is equal to production costs c,

$$p^* = M - (M-c) = \underline{\underline{c}}, \tag{1.23}$$

and the profit of particular firms is equal to zero,

$$u^* = \underline{\underline{0}}. \tag{1.24}$$

The following table summarizes the above results:

	Total Production q^*	Price p^*	Total Profit u^*
Monopoly	$\frac{1}{2}(M-c)$	$\frac{1}{2}M+\frac{1}{2}c$	$\frac{1}{4}(M-c)^2$
Duopoly	$\frac{2}{3}(M-c)$	$\frac{1}{3}M+\frac{2}{3}c$	$\frac{2}{9}(M-c)^2$
Oligopoly	$\frac{n}{n+1}(M-c)$	$\frac{1}{n+1}M+\frac{n}{n+1}c$	$\frac{n}{(n+1)^2}(M-c)^2$
Perfect Compet.	$(M-c)$	c	0