KAREL RYCHLÍK AND BERNARD BOLZANO

Towards the end of his life Bernard Bolzano sought a continuator of his mathematical work. Finally he invested his hopes to young Robert Zimmermann (1824 – 1898) to whom he then willed his mathematical manuscripts. But Robert Zimmermann concentrated only on philosophy and later became an associate professor (1849) and a professor of philosophy (in Prague since 1852, in Vienna since 1861) and a member of the Vienna Academy of Sciences. In 1882 he handed Bolzano's mathematical inheritance over to the Vienna Academy of Sciences and it passed it on, while Zimmermann was still alive, to the manuscript department of the Vienna Court Library, later National Library.

Very shortly after Bolzano's death M. J. Fesl, the good friend of Bernard Bolzano, attempted to publish Bolzano's whole work. His project was divided into twenty volumes and subdivided into five main groups. But it did not mature and the means accumulated for this purpose were only sufficient to publish four volumes of Bolzano's *Erbauungsreden* (Praha, 1848 – 1852). Not even Braumüller's edition called *Dr. Bernard Bolzano's gesammelte Schriften. Neue Ausgabe in zwölf Bände* (Wien, 1882) was successful, because it concentrated primarily on publishing Bolzano's treatise which had already been published and omitted the most important part of his mathematical heritage.[1]

At the end of the World War I Martin Jašek (1879 – 1945), secondary school teacher in Pilsen, looked into Bolzano's inheritance deposited in the Vienna National Library and found an extensive manuscript devoted to theory of functions. Soon he lectured on some interesting thoughts contained in this manuscript and he also wrote three reports on this topic.[2]

This discovery stimulated Czech Mathematicians to study and order Bolzano's inheritance. On 5 June 1924 the Bolzano Committee under the Royal Bohemian Society of Sciences was constituted. Members of this committee were K. Petr – chairman, M. Jašek – secretary, J. Sobotka, F. Krejčí, M. Horáček, B. Bydžovský, K. Vorovka, J. Vojtěch and Karel Rychlík.[3] The committee decided to make photocopies (so called "black snaps" – white letters on the black background) of the most important manuscripts deposited in the Vienna National Library. The Royal Bohemian Society of Sciences supported for this purpose M. Jašek, who stayed in Vienna studying Bolzano's mathematical manuscripts for two years and who also prepared those photocopies.

A lot of problems turned out to be much more difficult than they had seemed to be at the beginning. So publishing of Bolzano's manuscripts was delayed. In 1930 the Royal Bohemian Society of Sciences finally started to publish *Spisy Bernarda Bolzana (The Works*

[1] *Bernard Bolzano, Early Mathematical Works*, ed. L. Nový, in: Czechoslovak Studies in the History of Sciences, Special Issue 12, Prague 1981, pp. VIII–IX.

[2] *Aus dem handschriftlichen Nachlass Bernard Bolzanos. Das erste, historisch nachweisbare Beispiel einer stetigen nirgends differenzierbaren Funktion. Enthalten in dem Manuskripte Bolzanos "Functionenlehre" (Nationalbibliothek in Wien)* , in: Věstník Královské české společnosti nauk, Třída matematicko–přírodovědecká (1921), č. 1, pp. 1–32, also Jahresbericht der deutschen Matematiker–Vereinigung 31 (1922), 2. Abt., pp. 109–110;
Funkce Bolzanova, Časopis pro pěstování matematiky a fysiky 51 (1922), pp. 69–76;
O funkcích s nekonečným počtem oscilací v rukopisech Bernarda Bolzana, 53 (1923–24), pp. 102–109.

[3] Central Archives of the Academy of Sciences of Czech Republic, fund KČSN, carton 53, inventory number 292.

of Bernard Bolzano). But until the end of its existence only five volumes were published on the whole:

Vol. 1, *Functionenlehre* (K. Rychlík, ed.), Praha, 1930;
Vol. 2, *Zahlentheorie* (K. Rychlík, ed.), Praha, 1931;
Vol. 3, *Von dem besten Staate (G. Kowalewski, ed.)*, Praha, 1932;
Vol. 4, *Der Briefwechsel B. Bolzano's mit F. Exner (E. Winter, ed.)*, Praha, 1935;
Vol. 5, *Memoires géométriques (J. Vojtěch, ed.)*, Praha, 1948.

In 1952 the Bolzano Committee was abolished together with the Royal Bohemian Society of Sciences. At the same time the Czechoslovak Academy of Sciences was established, but the Bolzano Committee was restored only in 1958. Members of this new committee were M. Kössler – chairman, O. Borůvka, V. Kořínek, K. Rychlík, I. Seidlerová and J. Holubář.[4]

On 17 November 1968 Karel Rychlík was reworded in memoriam for his activities in the history of mathematics, above all for his scientific work on Bolzano's mathematical inheritance.

Now let us turn to Bolzano's manuscripts being studied by Karel Rychlík and to results of this Rychlík's work.

Functionenlehre

This manuscript is deposited in the National Library in Vienna as the 12th volume of the VIIth section called *Grössenlehre*. Photocopies of the manuscript are deposited in the Central Archives of the Czech Academy of Sciences in Prague;[5] some notes written by M. Jašek are enclosed.

The best known result of this manuscript is Bolzano's excellent example of a continuous function which has a derivative at no point of a certain everywhere dense set. (Now we know that this function has a derivative at no point at all, but Bolzano neither proved nor asserted this.) If we realise that the first example of a continuous function which has a derivative at no point was published in 1875 (see below), while Bolzano had written his work before the year 1834, then we understand the sensation caused by Jašek's report on the disclosure of this function, now known as *Bolzano's function*, in Bolzano's manuscript. Of course, the main significance of Functionenlehre consists not in this example but in a systematic exposition of the theory of continuity and derivative of functions of one variable.

The goal of this section is to describe the work of Karel Rychlík concerning Bernard Bolzano. For this purpose let us write something more about Bolzano's function. First it was constructed as an example of a continuous function whose (bounded) definition domain cannot be divided into a finite number of intervals of monotonicity. Bolzano shows that this function is continuous in the interval $[a,b]$, but it is monotone in no subinterval. Later Bolzano shows that the points at which $f(x)$ has no derivative are everywhere dense in the interval $[a,b]$. Even the fact that it occured to Bolzano at all that such a function might exist deserves our respect. The fact that he succeeded in actually constructing such a function is even more admirable.

Bolzano's function is defined as a limit of continuous functions $f_1(x),\ f_2(x),\ f_3(x),\ \dots$ defined in an interval $[a,b]$.

Here $f_1(x)$ is a function for which $f_1(a)=A$ and $f_1(b)=B$ and which is linear in the interval $[a,b]$:

$$f_1(x) = A + (x-a)\frac{B-A}{b-a}.$$

[4] Central Archives of the Academy of Sciences of Czech Republic, fund I. sekce ČSAV, carton 15, inventory number 38.

[5] Central Archives of the Czech Academy of Sciences, fund KČSN, carton 95, inventory number 624.

To define the function $f_2(x)$ Bolzano divides the interval $[a,b]$ into four subintervals limited by points:

$$a,\ a+\frac{3}{8}(b-a),\ \frac{1}{2}(a+b),\ a+\frac{7}{8}(b-a),\ b.$$

To these points Bolzano assigns values:

$$A,\ A+\frac{5}{8}(B-A),\ A+\frac{1}{2}(A+B),\ B+\frac{1}{8}(B-A),\ B;$$

withal the function $f_2(x)$ is linear in each of the four subintervals.

The function $f_3(x)$ is defined analogously, only every of the four subinterval is considered instead of the interval $[a,b]$, etc.

Evidently, if the limit

$$f(x)=\lim_{n\to\infty} f_n(x) \tag{2.1}$$

exists, then $f(x)$ is monotone in no subinterval of $[a,b]$. Indeed, at all vertices of the curve $f_n(x)$ we have obviously

$$f(x)=f_n(x);$$

those vertices of the curve, at which the curve $f_n(x)$ changes from an ascending one to a descending one or vice versa, are distributed in the interval $[a,b]$ for n sufficiently large as densely as we desire. So far Bolzano's reasoning is correct.

Still it remains to be proved that the limit (2.1) exists and is a continuous function in the interval $[a,b]$. Bolzano's proof of the existence of this limit is essentially correct. On the other hand, he commits a substantial error when proving the continuity of the function $f(x)$. He argues simply as follows: the function $f(x)$ is a limit of continuous functions $f_n(x)$, and the limit of a convergent sequence of continuous functions is always a continuous function. Of course the last assertion is wrong. It becomes true, however, if we require for example *uniform* convergence.

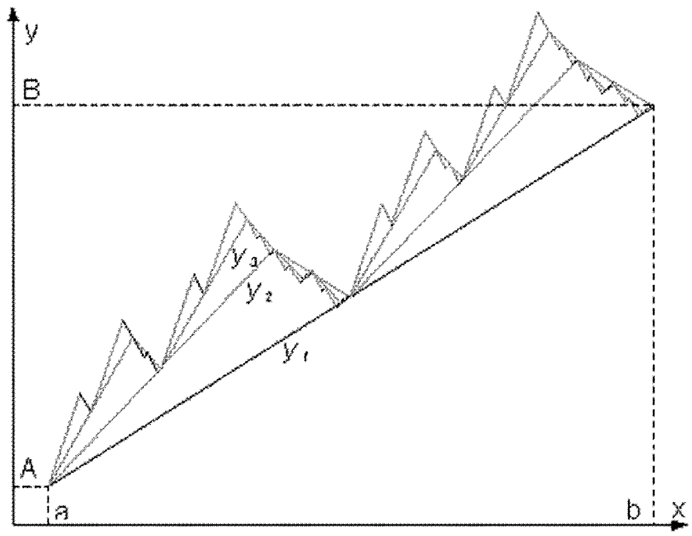


FIGURE 2.1 BOLZANO'S FUNCTION.

Notation:
$y_1(x) = f_1(x)$,
$y_2(x) = f_2(x)$,
$y_3(x) = f_3(x)$,
etc.

The first report of M. Jašek on Bolzano's *Functionenlehre* was held on 3 December 1921. Already on 2 February 1922 Karel Rychlík introduced to the Royal Bohemian Society of Sciences his treatise *Über eine Funktion aus Bolzanos handschriftlichen Nachlasse.*[6] On 9 February 1922 Rychlík lectured on the same topic in Society of Czechoslovak Mathematicians and Physicists. He gave the correct proof of continuity of Bolzano's function and he proved a strong assertion that this function has a derivative at no point of the interval (a, b) (finite nor infinite).

The same assertion was proved by Vojtěch Jarník in the same time but in a different way. His proof is described in the treatise *O funkci Bolzanově.*[7]

Later Karel Rychlík lectured on *Functionenlehre* twice more:

O Bolzanově nauce o funkcích (12. 5. 1927, Brno, Society of Czechoslovak Mathematicians and Physicists) and
La théorie de fonction de Bolzano (3. –10. 9. 1928, Bologna, The International Congress of Mathematicians); the treatise with the same title was published in the congress memorial volume.

In 1930 *Functionenlehre* was finally published. It was edited by Karel Rychlík and it was the first volume of *Spisy Bernarda Bolzana.* The book is provided with careful, detailed notes by Karel Rychlík and with an interesting foreword of Karel Petr.

Let us close the section devoted to Bolzano's *Functionenlehre* with words of Vojtěch Jarník:

It is such an extraordinary work that we cannot but regret that, as an unpublished manuscript, it had not the opportunity to influence the development of mathematics in his own time. In Bolzano's days ... the theory of functions was already considerably developed, its main concepts, however, lacked sharp contours and the principal theorems were not upheld by exact proofs. And it is in the very foundations of the theory of functions that Bolzano's Functionelehre represents a virtual milestone, unfortunately a milestone overgrown with the moss of ignorance.

Among Bolzano's contemporaries, only Gauss, Abel and Cauchy manifested the same sense for the proper construction of the foundations of the theory of functions. Two of them, Gauss and Abel, presented masterpieces of exact mathematical methods but did not deal with these fundamental problems systematically. The last of them, Cauchy, in his works

[6]Věstník Královské české společnosti nauk, Třída matematicko–přírodovědecká (1921–22), č. IV, 20 pages.
[7]Časopis pro pěstování matematiky a fysiky 51 (1922), pp. 248–264.

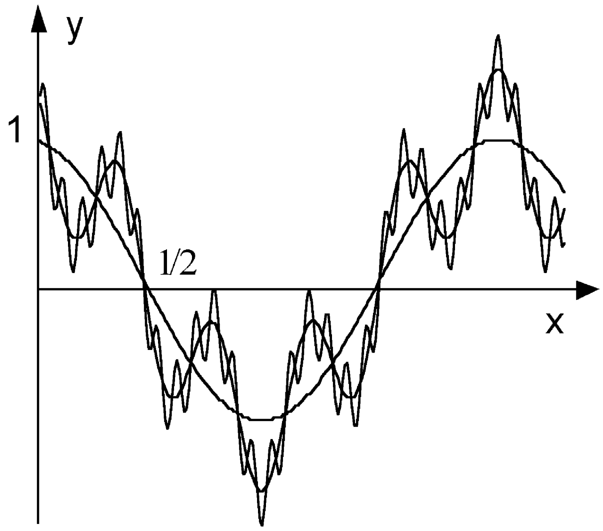
Cours d'Analyse (1821), Résumé des leçons ... sur le Calcul Infinitésimal (1823), Leçons sur le Calcul différentiel (1829) based the main branches of the theory of functions ... on firm foundations (or let us say more carefully on firmer foundations) in a systematic way. However, Bolzano goes in his efforts even beyond Cauchy's achievements. Cauchy usually contented himself with building the foundations to a level necessary for his further deductions; unlike him, Bolzano was more of a philosopher, interested in the fundamental problems of mathematics. We shall see later how rigorously Bolzano introduces his definitions, how critically he dissects his concepts, with what deep interest and thoroughness he discusses all logically possible cases regardless of their greater or lesser importance for concrete mathematical problems ... .[8]

[8] Vojtěch Jarník: *Bolzano and the foundations of mathematical analysis*, Praha, 1981.

Continuous Nowhere Differentiable Functions

For deeper understanding the extraordinarity of Bolzano's function let us describe the history of continuous nowhere differentiable functions. Recall that the Bolzano's manuscript had been finished before the year 1834!

On 18 July 1872 Karl Weierstrass lectured in the Royal Academy of Sciences in Berlin on the function which is continuous in the domain of all real numbers but it has derivative an no real point:



Three approximations of Weierstrass' function for $a = 1/2,\ b = 5$.

FIGURE 2.2 WEIERSTRASS' FUNCTION.

This example is defined as follows:

$$f(x) = \sum_{n=1}^{\infty} a^n \cos(\pi b^n x);$$

$$0 < a < 1;\ ab > 1 + \frac{3}{2}\pi.$$

Weierstrass' function was published in 1875 by P. du Bois–Reymond, the student of Karl Weierstrass, in the treatise *Versuch einer Classification der willkürlichen Functionen reeler Argumente nach ihren Aenderungen in den kleinsten Intervallen.*[9] Of course, du Bois–Reymond quoted Weierstrass' name. Weierstrass himself published his example only in 1880.

For a long time Weierstrass' example was beeing considered as the first example of the continuous nowhere differentiable function. Since then many mathematicians were interested in this topic:
G. Darboux, *Mémoire sur les fonctions discontinues*,[10]
V. Dini, *Fondamenti per la teorica delle funzioni di variabili reali*,[11]
M. Lerch, *Über die Nichtdifferentiirbarkeit gewisser Function*,[12]
and others.

[9] Journal für die reine und angewandte Mathematik 79(1875), pp. 21–37.
[10] Annales de l'Ecole normale (2) IV (1875), pp. 57–112.
[11] Pisa, 1878.
[12] Journal für die reine und angewandte Mathematik 92 (1888), pp. 126–138.

In 1890 the example constructed by Ch. Cellèrier already in 1860 was posthumously published in the paper *Note sur les principes fondamentaux de l'analyse.*[13]

Cellèrier's function is defined alike the Weierstrass' one:

$$f(x) = \sum_{n=0}^{n} b^{-n} \sin(\pi b^n x);\; b > 1000.$$

The fact that it was written already in 1860 caused the real sensation. We can imagine the sensation caused by Jašek's disclosure of Bolzano's function which was constructed before the year 1834.

Let us continue in the survey of the history. In 1903 the function constructed by T. Takagi was published in the treatise *A simple example of the continuous function without derivative*:[14]

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \Delta(2^n x);\; \Delta(x) = dist(x, \mathbb{Z}).$$

The method of the construction of this example is recognizable from Figure 2.3:

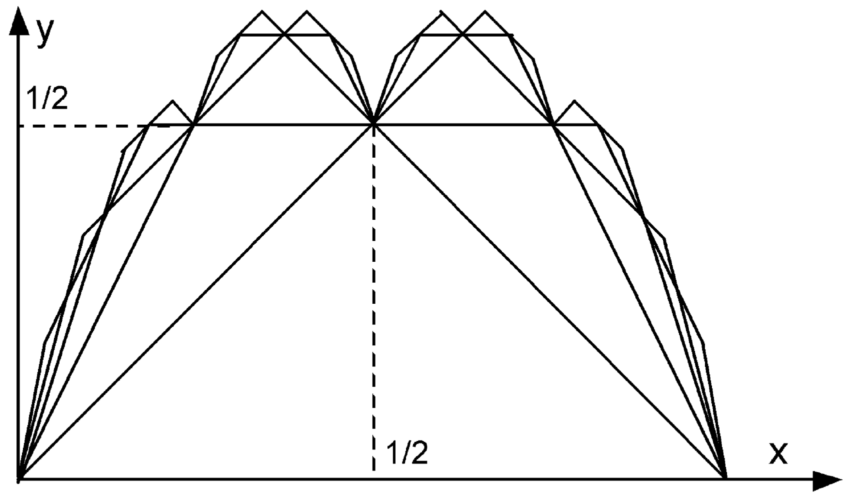


FIGURE 2.3 TAKAGI'S FUNCTION.

It's modification is now known as the so–called *Waerden's example. It was published in 1930 by B. L. van der Waerden in his paper Ein einfaches Beispiel einer nicht-differenzierbaren stetigen Funktion.*[15] Waerden's example is generally considered to be the easiest example of the continuous function without derivative.

In 1920 the Czech mathematician Karel Petr published in the Czech journal *Časopis pro pěstování matematiky a fysiky* his example of continuous function without derivative. This function is not quoted as common as the ones described above, bud it is very simple. Only the knowledge of the definition of continuity and derivative and a simple aritmetic theorem is necessary to understand both the construction and the proof of continuity and non–differentiability.

Petr's function is defined in the interval $[0, 1]$ as follows:

[13]Bulletin des sciences mathématiques 14 (1890), pp. 142–160.
[14]Tokyo sugaku butsurigaku kwai kiji (English and Japan) 1 (1903), pp. 176–177.
[15]Mathematische Zeitschrift 32 (1930), pp. 474–475.

$$x = \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \frac{a_4}{10^4} + \ldots;\ a_k \text{ integers; } 0 \le a_k \le 9;$$

$$f(x) = \frac{b_1}{2^1} \pm \frac{b_2}{2^2} \pm \frac{b_3}{2^3} \pm \frac{b_4}{2^4} \pm \ldots;\ b_k = \begin{cases} 0 & \text{for even } a_k \\ 1 & \text{for odd } a_k \end{cases}$$

$$\text{sign before } b_{k+1} \begin{cases} \text{opposite than the one before } b_k \text{ for } a_k \in \{1,\ 3,\ 5,\ 7\} \\ \text{thesame otherwise} \end{cases}$$

In Figure 2.4 you can see the chart of an approximation of the Petr's function. In order to show it more graphically a four–adic number system is used. If you compare both charts bellow, you will understand the necessity of the exception to the rule of sign assignment awarded to the digit 9; the result would not be a continuous function:

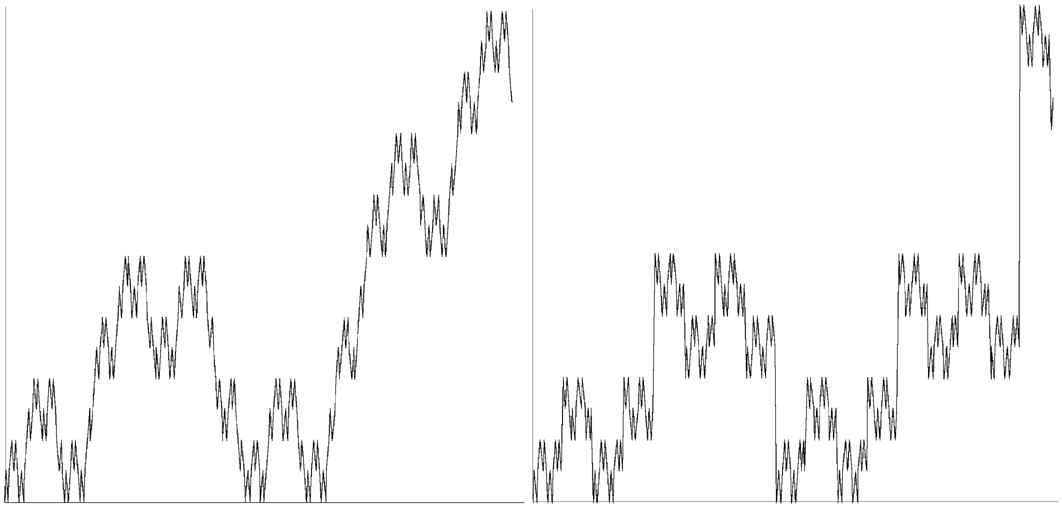


FIGURE 2.4 PETR'S FUNCTION.

The same year and in the same journal Karel Rychlík generalized Petr's function in his paper *Funkce spojité nemající pro žádnou hodnotu proměnné derivace v tělese čísel Henselových.*[16] The German version called *Eine stetige nicht differenzierbare Funktion im Gebiete der Henselschen Zahlen*[17] was published two years later. Karel Rychlík constructed his function in the field of *p–adic numbers*, named after Kurt Hensel who introduced them into mathematics (today the denomination *Hensel's numbers* is not common).

Every element of the p–adic number field can be expressed as follows:

$$x = a_r p^r + a_{r+1} p^{r+1} + \ldots,$$

where r is an integer and coefficients a_i are elements of the set $0, 1, \ldots\ p-1$.

The following value is assigned to this p–adic number:

$$f(x) = a_r p^r + a_{r+2} p^{r+2} + a_{r+4} p^{r+4} + \ldots\ .$$

[16] Časopis pro pěstování matematiky a fyziky 49 (1920), pp. 222–223.

[17] Journal für die reine und angew. Mathematik 152 (1923), pp. 178–179.

Zahlenlehre

The manuscript of this Bolzano's work is deposited in the National Library in Vienna as the 8th, 9th and 10th volume (the first, second and third version) of the VIIth section of the Bolzano's inheritance. Photocopies of the whole third version and several demonstrations of the previous ones are deposited in the Central Archives of the Czech Academy of Sciences in Prague.[18] These photocopies are divided into 8 parts being named *Zahlenlehre I–VIII*; some introductory notes written by M. Jašek are added.

Zahlenlehre is again the part of the extensive work *Grössenlehre*; Bolzano was working on this manuscript from 1830 to 1835.[19]

Zahlenlehre IV

This part of extensive *Zahlenlehre,* denoted by Bolzano himself *Zweyter Abschnitt: Verhältniss der Theilbarkeit unter den Zahlen*, was published in 1931. The book is entitled *Zahlentheorie*, it forms the second volume of *Spisy Bernarda Bolzana* (see above) and it was Karel Rychlík who had prepared the manuscript for printing.

The manuscript treats elementary properties of natural numbers, being called (by Bernard Bolzano) *wirkliche Zahlen* – true numbers. Alike *Functionenlehre*, also *Zahlentheorie* is provided with careful and detailed critical notes by Karel Rychlík. Some of these notes precise or correct Bolzano's assertions. For example Bolzano states a wrong generalisation of the auxiliary Lagrange theorem necessary for proving the following theorem:

Every positive number (understand integer) *can be expressed as the sum of at the most four non zero quadrates.*

Rychlík corrects Bolzano's proof of that auxiliary Lagrange theorem. This correction is made in the most carefully way; Rychlík preserves basic Bolzano's ideas.

In other notes Rychlík gives a version of a theorem common in modern mathematics, generalisations (for example he generalises the equation $ax + by = d$, where d is the greatest common divisor of numbers a and b, to the case of l numbers) or strenghtens some theorems (for example Bolzano proves: $[a/b] \geq n[a/nb]$; Rychlík states: $[[a/b]/n] = [a/nb]$).

Moreover there are many historical notes.

On 6 March 1930 Karel Rychlík lectured on Bolzano's *Zahlentheorie* in the Society of Czechoslovak Mathematicians and Physicists. The lecture was named *O Bolzanově číselné theorii.*

Zahlenlehre II

At the beginning of fifties Karel Rychlík started to work on this part of Bolzano's *Zahlenlehre.* In 1958 it was almost ready for printing.[20] The work was finally published in 1962 and it was entitled *Theorie der reelen Zahlen im Bolzanos handschriftlichen Nachlasse.* The book is provided with Rychlík's foreword, introduction, concluding notes and a survey of history of real numbers. The book is also equipped with a foreword by Ladislav Rieger.

This part of Bolzano's manuscript *Zahlenlehre* is inherent only in the second and the third versions; Bolzano himself called it *Unendliche Grössenbegriffe.*

In his notes Karel Rychlík gives a possible interpretation of Bolzano's theory, which is not correct. He tries to preserve as most as possible. He assigns these concepts using the Cantor theory of real numbers:

[18] Central Archives of the Czech Academy of Sciences, fund KČSN, carton 93, 94, inventory numbers 616–623.

[19] E. Winter: B. Bolzano und sein Kreis, Lipsko 1933, kap. V.

[20] According to the protocol of the session of the Bolzano Committee being held on 24 October 1958; Central Archives of the Czech Academy of Sciences, fund I. sekce ČSAV 1952–1961, carton 15, inventory number 38.

in Bolzano's theory	in Rychlík's interpretation
infinite number expression	*sequence of rational numbers*
mensurable number	*convergent sequence of rational numbers*
equality of mensurable numbers	*equivalence of convergent sequences*

Here an *infinite number expression* is the term for an expression where an infinite number of operations (addition, subtraction, multiplication and division) with integers is.

An *mensurable number* denotes such an expression S that for each positive integer q exists an integer p for which these equalities hold:

$$S = \frac{p}{q} + P_1; \qquad S = \frac{p+1}{q} - P_2,$$

where P_1, (P_2) is a non–negative (positive) number expression.

Finally *mensurable numbers A, B are equal* if and only if for each positive integer q there exists an integer p such that:

$$A = \frac{p}{q} + P_1 = \frac{p+1}{q} - P_2; \; B = \frac{p}{q} + P_3 = \frac{p+1}{q} - P_4,$$

where P_1, P_3 (P_2, P_4) are non–negative (positive) expressions.

Before the book was published, Rychlík reported on its content as well as on his own interpretation in four treatises:

Teorie reálných čísel v Bolzanově rukopisné pozůstalosti,[21]
Theorie der reellen Zahlen im Bolzanos handschriftlichen Nachlasse,[22]
Těorija věščestvěnnych čisel v rukopisnom nasledii Bolzano,[23] and
La théorie des nombres reéles daus un ouvrage posthume manuscript de B. Bolzano.[24]

Karel Rychlík also lectured on this Bolzano's manuscript in the lecture *Teorie reálných čísel z Bolzanovy rukopisné pozůstalosti*,which was held on 25 November 1957 in Prague.

Other Parts of Zahlenlehre

Photocopies of sections IV and VI–VIII are equipped with notes written by Karel Rychlík in March 1951. In these notes Rychlík describes a content of respective parts of manuscript and explanes which versions they are contained in. Other sections are sometimes provided with Rychlík's notes written on the title–page of subsections.

Until October 1958 Karel Rychlík had rewritten *Zahlenlehre I* and *II* (see above) and the transcript of *Zahlenlehre III* was semi–finished by that time.[25]

Versuch einer Erklärung der Begriffe von Linie, Fläche und Körper

The original manuscript is deposited in the National Library in Vienna in the VIth section named *Zu vier problemen der Geometrie und Anti-Euklid* consisting of five volumes. It is the third part of the third volume named *Zu den Problemen der Rectification, der Complanation und der Cubierung.*[26] The overwrite taken by Robert Zimmermann is deposited

[21]Časopis pro pěstování matematiky 81 (1956), pp. 391–395.
[22]Čechoslovackij matěmatičeskij žurnal 7 (82)(1957), pp. 553–567.
[23]Istor.-matěm. issledovanija 11 (1958), pp. 231–245.
[24]Revue d'histoire sciences 14 (1961), pp. 313–327.
[25]According to the protocol of the session of the Bolzano Committee being held in October 1958; Central Archives of the Czech Academy of Sciences, fund I. sekce ČSAV 1952–1961, carton 15, inventory number 38.
[26]The fourth volume of the VIth section named *Über Haltung, Richtung, Krümmung und Schnörkelung bei Linien, sowohl als Flächen samt einigen verwandten Begriffen* was published in 1948 as the fifth volume of *Spisy Bernarda Bolzana* being entitled *Geometrische Arbeiten* (see above).

in the Literary Archives in Prague.[27] Photocopies of the original provided by M. Jašek are deposited in the Central Archives of the Czech Academy of Sciences in Prague.[28]

Karel Rychlík rewrote the Zimmermann's version of Bolzano's work comparing it with photocopies of the original manuscript. He handed the transcription consisting of 41 typewriting pages over to the Royal Bohemian Society in June 1951. Rychlík's overwrite has remained being deposited in the archives;[29] no Rychlík's publication concerns this manuscript.

Von der Mathematischen Lehrart

Von der Mathematischen Lehrart constitutes the second part of the manuscript named *Einleitende Teile* which is deposited in the National Library in Vienna and which forms fourth, fifth and sixth volumes of the VIIth section (accordant with first, second and third versions of the same manuscript). The first part is named *Von dem Begriffe der Mathematik und ihren Teilen*, the third one is entitled *Vorkenntnisse.* Photocopies of the second part are deposited in the Central Archives of the Czech Academy of Sciences in Prague;[30] they are provided with notions written by M. Jašek. According to these notions Jašek had taken photocopies of the whole third version being written by an amanuensis and corrected by Bernard Bolzano himself (moreover many pages are autographical by Bolzano). There are only photocopies of the second part (the third version and several demonstrations of previous ones) in the archives.

The third version of the second part was overwritten by Ferdinand Pelikán in 1951.[31] The same year Karel Rychlík collated it[32] and in 1958 he even started to prepare this work for printing.[33]

Two Rychlík's treatises concern Bolzano's manuscript *Von der mathematischen Lehrart.* Both were published in 1958:

Úvahy z logiky v Bolzanově rukopisné pozůstalosti[34] and
Betrachtungen aus der Logik in Bolzanos handschriftlichen Nachlasse.[35]

The content of both papers is the same. Rychlík gives the résumé of Bolzano's manuscript and a demonstration (section 8). Bozlano drew up this work as the second part of the introduction to *Grössenlehre.* It is devoted above all to logic.

Zur Mathematik

The manuscript is deposited in the National Library in Vienna and it forms the first volume of the VIIth section of Bolzano's inheritance. Its photocopies provided by M. Jašek are deposited again in the Central Archives of the Czech Academy of Sciences.[36] The photocopies are devided into portfolios being inscribed by Karel Rychlík. In accordance with this fact Rychlík at least saw it through and assorted it. Date notes are missing so that we cannot say when Rychlík was dealing with this manuscript.

[27] Literary Archives, fund P4a, carton Y, 2811, 7 copy–books.

[28] Central Archives of the Czech Academy of Sciences, fund KČSN, carton 92, inventory number 613.

[29] Central Archives of the Czech Academy of Sciences, fund KČSN, carton 91, inventory number 609.

[30] Central Archives of the Czech Academy of Sciences, fund KČSN, carton 92, inventory number 615.

[31] Central Archives of the Czech Academy of Sciences, fund KČSN, carton 91, inventory number 610; this work consists of 60 typewriting pages.

[32] It means he compared it with the original.

[33] Central Archives of the Czech Academy of Sciences, fund I. sekce ČSAV 1952–1961, carton 15, inventory number 38.

[34] Časopis pro pěstování matematiky a fyziky 83 (1958), pp. 230–235

[35] Čechoslovackij matěmatičeskij žurnal 8 (83) (1958), pp. 197–202

[36] Central Archives of the Czech Academy of Sciences, fund KČSN, carton 92, inventory number 614.

Zeit– und Raumlehre

This manuscript constitutes the 14th (and the last) volume of the VIIth section of Bolzano's inheritance deposited in the National Library in Vienna. Its photocopies are deposited in the Central Archives of the Czech Academy of Sciences.[37] Alike the previous ones these photocopies are devided into portfolios being inscribed by Karel Rychlík. Moreover there are notes written by Karel Rychlík and dated in March 1951.

Other Works

Besides treatises quoted above Karel Rychlík devoted to Bernard Bolzano two more papers:

Bolzanův pobyt v Liběchově[38] and
Sur les contacts personnels de Cauchy et de Bolzano.[39]

The first work describes Bolzano's stay at the manor of A. Vith in Liběchov in 1847. Here Bernard Bolzano started writing his famous *Paradoxien des Unendlichen.*

The second one discusses the possibility of get-together of B. Bolzano and A. L. Cauchy. First Rychlík quotes the treatise written by Ruth and Dirk J. Struik in *Isis* in 1928. They assume that the meeting is implausible. Then Rychlík quotes the rewiew of E. Winter's book *Der böhmische Vormärz in Briefen B. Bolzanos an F. Příhonský (1924–1848)*[40] written by P. Funk in 1957.[41] Funk inclines to the positive answer, but his argument is not completely satisfactory. But I. Seidlerová had found a letter of Bolzano to Fesl to Vienna which is together with the rest of their correspondence deposited in the Literary Archives in Prague. She tould Rychlík about it and she also published a paper discussing this letter. The thing is, that Bolzano wrote to Fesl that he had met Cauchy during his stay in Prague several times. And this can be considered as the proof of the assumption that Bolzano and Cauchy encountered.

Figure 2.5 Bernard Bolzano.

[37]Central Archives of the Czech Academy of Sciences, fund KČSN, carton 95, inventory number 625.
[38]Matematika 9 (1959), pp. 111–113.
[39]Revue d'histoire sciences 15 (1962), pp. 163–164.
[40]Berlín, 1956.
[41]Monatshefte für Mathematik 61, 1957, str. 251.