Analytical Design of FIR Filters

Miroslav Vlček, Pavel Zahradník and Rolf Unbehauen

***Abstract*— A new recursive algorithm for the impulse response coefficients of a FIR lowpass filter is developed. The algorithm is obtained from the differential equation for the amplitude response of a lowpass filter. While the original filter exhibits maximally flat frequency response, the abridging of the impulse response provides a frequency response comparable to those obtained by other design methods.**

***Index Terms*— FIR lowpass filter, impulse response, recurrence formula.**

I. Introduction

THE design of an arbitrary FIR filter seems as an almost closed chapter in digital signal processing. A number of sophisticated procedures and refined methods for design of linear phase FIR filters are recently available [1], [2], [3], [5], [6]. A preferable procedure is the McClellan-Parks program [3] which is used in the design of various equiripple FIR filters. Besides, there exists a group of methods attributed to the analytic design procedures. They are solely devoted to filters with maximally flat frequency response. Herrmann [2] obtained analytic formulae for the impulse response coefficients of maximally flat FIR filters using Bernstein polynomials. He deduced also an empirical formula for the degree estimation. Rajagopal and Dutta Roy [5] confirmed Herrmann's results by a firm theoretical examination. Cooklev and Nishihara [1] gave a generalization of the Bernstein polynomials for all four FIR linear filter cases. The order of the maximally flat FIR filters is substantially higher than the order of their equiripple counterparts and it consequently means that the number of multiplications required for an output sample is quite large. In that view, Vlček and Jireš [8] studied FIR notch filters and their abridging. They also emphasized that the simple abridging of a long maximally flat FIR filter produces a filter comparable to that obtained by a standard windowing technique.

In our paper we present a new recursive algorithm for the impulse response coefficients of a maximally flat FIR filter. The analytic design procedure is based on the first derivative of the pseudoamplitude $Q(w)$ of an FIR filter. We call this derivative the generating function. By the first derivative, any distribution of the minima and maxima of the pseudoamplitude maps to a set of real zeros of the generating function. The zeros are distributed over the two disjoint intervals which belong to the passband and stopband, respectively. Emphasizing the distribution of zeros for a maximally flat frequency response we have derived the differential equation for the generating function. The solution of the differential equation is found in form of the backward recursive algorithm for the impulse response coefficients.

II. Transfer Function and Generating Function

Due to the symmetry of the impulse response coefficients the transfer function of a linear phase FIR filter of length $N = 2M + 1$

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}\,, \tag{1}$$

can be written in form

$$H(z) = z^{-M}\left[h(M) + 2\sum_{m=1}^{M} h(M-m)\,\frac{1}{2}\left(z^{m} + z^{-m}\right)\right]. \tag{2}$$

Introducing into (2) Chebyshev polynomials of the first kind $T_m(w)$ and substituting

$$a(0) = h(M) \qquad a(m) = 2h(M-m) \tag{3}$$

we obtain

$$H(z) = z^{-M}\left[a(0) + \sum_{m=1}^{M} a(m)\,T_m(w)\right] = z^{-M} Q(w)\,. \tag{4}$$

In equation (4) $Q(w)$ represents the pseudoamplitude which for $w = \frac{1}{2}(z + z^{-1})\big|_{z=e^{j\omega T}} = cos\,\omega T$ reduces to a real valued frequency response of the zero-phase FIR filter. For convenience in notation we often refer to $a(m)$ as the impulse response coefficients understanding that true causal impulse response coefficients are obtained through a time shift. For a maximal ripple lowpass FIR filter as classified in [4] the pseudoamplitude $Q(w)$

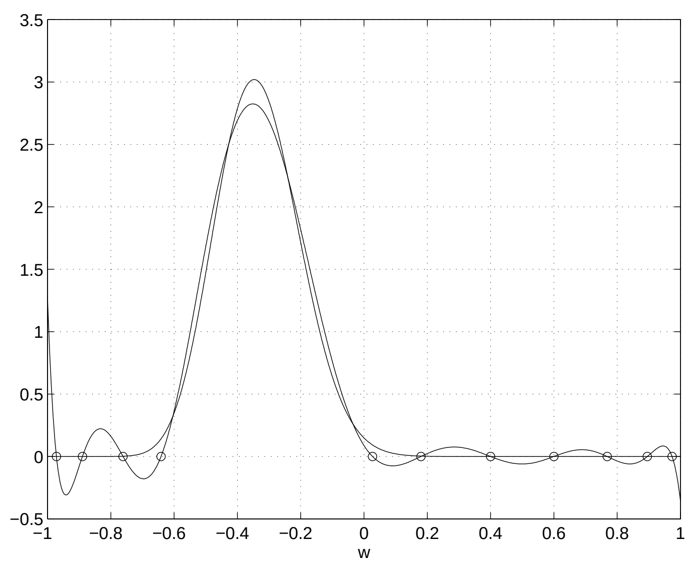

Fig. 1. $Q'(w)$ for optimal equiripple FIR filter of degree N = 25 and maximally flat FIR filter of degree N = 101.

$$Q(w) = a(0) + \sum_{m=1}^{M} a(m)\,T_m(w) \tag{5}$$

has alternating local minima and maxima distributed over the stopband $(-1,\ -w_p)$ and passband $(w_p,\ 1)$, respectively.

It means that the first derivative of the quantity $Q(w)$ as shown in Fig. 1

$$\frac{d}{dw}Q(w) = \sum_{m=1}^{M} m\ a(m)\ U_{m-1}(w) \tag{6}$$

has real zeros within these two disjoint intervals only and it is expressed by a sum of Chebyshev polynomials of the second kind $U_m(w)$ because the following identity holds

$$\frac{d}{dw}T_m(w) = \ m\ U_{m-1}(w)\,. \tag{7}$$

The equiripple approximation has the natural limit when the distribution of all zeros $w_{0\mu}$ from Fig. 1 is simplified to the set of zeros which are either confluent at 1 or at -1. Then the above equation (6) reduces to

$$\begin{aligned}\frac{d}{dw}Q(w) \equiv \frac{d}{dw}C_{p,q}(w) = \\ = 2^{-(p+q+1)}\ (p+q+1)\begin{pmatrix} p+q \\ p \end{pmatrix}(1-w)^p(1+w)^q\end{aligned} \tag{8}$$

and it represents the first derivative of a maximally flat frequency response which can be found in form of Bernstein polynomials [2], [5] as

$$C_{p,q}(w) = \left(\frac{1+w}{2}\right)^{q+1}\sum_{m=0}^{p}\begin{pmatrix} m+q \\ m \end{pmatrix}\left(\frac{1-w}{2}\right)^m. \tag{9}$$

We will call the first derivative of the pseudoamplitude $Q(w)$ a normalized generating function. For a maximally flat FIR filters it has the form of (8) in which p and q represent the order of flatness at $\omega = 0$ and $\omega = \pi$, respectively. They define the degree of a filter

$$M = p + q + 1\,. \tag{10}$$

III. Differential Equation and the Recursive Evaluation of the Impulse Response Coefficients

By differentiating (8) we get the differential equation

$$(1-w^2)\frac{d^2}{dw^2}C_{p,q}(w) + [p-q+(p+q)w]\,\frac{d}{dw}C_{p,q}(w) = 0 \tag{11}$$

which is the central result in our design procedure. As the general identity (6) holds we obtain the expression

$$\frac{d}{dw}Q(w) \equiv \frac{d}{dw}C_{p,q}(w) = \sum_{m=1}^{M}\ m\ a(m)\ U_{m-1}(w)\,, \tag{12}$$

which can be substituted into the differential equation (11) in order to find relations among the unknown values of the impulse response coefficients $a(m)$. By introducing the substitution

$$\alpha(k) = ka(k) \tag{13}$$

we obtain

$$\begin{aligned}(1-w^2)\frac{d^2}{dw^2}C_{p,q}(w) = \\ = \sum_{m=1}^{M}\alpha(m)(1-w^2)\,\frac{d}{dw}\,U_{m-1}(w) \\ = -\sum_{m=1}^{M}\frac{m-1}{2}\ \alpha(m)\,U_m(w) \\ + \sum_{m=1}^{M}\frac{m+1}{2}\ \alpha(m)\,U_{m-2}(w)\,.\end{aligned} \tag{14}$$

Substituting expression (14) into the differential equation (11) we can write it as

$$\begin{aligned}&\sum_{m=1}^{M}\frac{p+q+1-m}{2}\alpha(m)\,U_m(w) \\ +&\sum_{m=1}^{M}(p-q)\alpha(m)\,U_{m-1}(w) \\ +&\sum_{m=1}^{M}\frac{p+q+1+m}{2}\alpha(m)\,U_{m-2}(w) = 0\,.\end{aligned} \tag{15}$$

It should be satisfied for any power of the variable w. Since $\alpha(0) = 0$ and $U_{-1}(w) = 0$ we can rearrange the equation (15) into the form

$$\begin{aligned}&\sum_{m=1}^{M+1}\frac{p+q+2-m}{2}\alpha(m-1)\,U_{m-1}(w) \\ +&\sum_{m=1}^{M}(p-q)\alpha(m)\,U_{m-1}(w) \\ +&\sum_{m=1}^{M-1}\frac{p+q+2+m}{2}\alpha(m+1)\,U_{m-1}(w) = 0\,.\end{aligned} \tag{16}$$

Then the identity (16) provides a backward recursive evaluation of the impulse response coefficients. If $m = M+1$, the first sum in (16) produces

$$\frac{p+q+1-M}{2}\alpha(M)\ U_M(w) = 0 \tag{17}$$

which can be satisfied by equation (10) only. Value $m = M$ provides

$$\left(\frac{p+q+2-M}{2}\alpha(M-1)\ + (p-q)\alpha(M)\right)U_{M-1}(w) = 0 \tag{18}$$

or

$$\alpha(M-1)\ = -2(p-q)\alpha(M)\,, \tag{19}$$

which is the initial condition for the backward recursion. Comparing the expression (6) with (8) we conclude that the highest power of w^{p+q} is accompanied by a coefficient $(-1)^pMa(M)2^{p+q}$. The factor 2^{p+q} appears here due to the Chebyshev polynomial expansion. It consequently gives the initial value of coefficient $\alpha(M)$ needed in (18) as

$$\alpha(M) = \frac{1}{2}(-1)^p2^{-2(p+q)}M\begin{pmatrix} p+q \\ p \end{pmatrix}. \tag{20}$$

The general backward recursion is based on a general identity following from (16) which is concisely expressed by the algorithm summarized in the Tab. I.

From the algorithm we are able to evaluate all the impulse response coefficients except the $a(0)$. This coefficient is obtained from the unit value of $Q(w=1) \equiv 1$,

$$C_{p,q}(1) = a(0) + \sum_{m=1}^{M}\ a(m) = 1\,. \tag{21}$$

So that

$$a(0) = 1 - \sum_{m=1}^{M}\ a(m)\,. \tag{22}$$

TABLE I

BACKWARD RECURSION FOR THE IMPULSE RESPONSE COEFFICIENTS.

given	p, q
initialization	$\alpha(M) = \frac{1}{2}(-1)^p 2^{-2(p+q)} M \begin{pmatrix} p+q \\ p \end{pmatrix}$
	$\alpha(M-1) = -2(p-q)\alpha(M)$
body	
(*for* $k = M-1$ *to* 2)	
	$\frac{M+1-k}{2}\alpha(k-1) = (q-p)\alpha(k) - \frac{M+1+k}{2}\alpha(k+1)$
(*end loop on* k)	
integration	
(*for* $k = M$ *to* 1)	
	$a(k) = \frac{\alpha(k)}{k}$
(*end loop on* k)	

This recursion completes the analytic procedure of direct design of FIR maximally flat filters. Note that the whole design process is a recursive one and it does not require any DFT algorithm nor we need any iterative technique.

The main disadvantage of these filters is that the estimated filter order is approximately inversely proportional to the square of the transition bandwidth [7]. The design procedure usually leads to filters of much higher order than those with equiripple frequency response. The economization of Chebyshev polynomial expansion of the transfer function is equivalent to the square windowing of a finite but large extent impulse response [8]. For the maximally flat pseudoamplitude we can write

$$\begin{aligned} Q(w) &= a(0) + \sum_{m=1}^{L} a(m)\, T_m(w) + \sum_{m=L+1}^{M} a(m)\, T_m(w) \\ &= Q_r(w) + \sum_{m=L+1}^{M} a(m)\, T_m(w) \,. \end{aligned} \tag{23}$$

Since $|T_m(w)| \leq 1$, the reduced pseudoamplitude $Q_r(w)$ satisfies the required filter specifications within the accuracy

$$\delta_p, \delta_s \leq \ \delta_e \ = \sum_{m=L+1}^{M} |a(m)| \,. \tag{24}$$

IV. FILTER DESIGN, ABRIDGING

Despite of the fact that the degree equation is not available, the parameters of the abridged filters w_p, w_s, δ_p and δ_s can be estimated as a function of M, L, p and q. From the differential equation (11) the maximum of generating function (8) is found

$$w_m = \frac{q-p}{q+p} \ . \tag{25}$$

From (8) and (9) follows

$$\begin{aligned} &C'_{p,q}(w_m) = \\ &\quad = 2^{-(p+q+1)}(p+q+1) \begin{pmatrix} p+q \\ p \end{pmatrix} \left(\frac{2p}{q+p}\right)^p \left(\frac{2q}{q+p}\right)^q \end{aligned} \tag{26}$$

and

$$C_{p,q}(w_m) = \left(\frac{q}{q+p}\right)^{q+1} \sum_{m=0}^{p} \begin{pmatrix} q+m \\ m \end{pmatrix} \left(\frac{p}{q+p}\right)^m . \tag{27}$$

The estimated passband and stopband edges in the w-domain are given by

$$w_p \leq w_{ep} = \frac{1 - C_{p,q}(w_m) + w_m C'_{p,q}(w_m)}{C'_{p,q}(w_m)} \tag{28}$$

$$w_{es} = -\frac{C_{p,q}(w_m) - w_m C'_{p,q}(w_m)}{C'_{p,q}(w_m)} \leq w_s \tag{29}$$

which are related to the frequency domain by relations

$$w_{ep} = \cos \omega_{ep} T \ \ , \quad w_{es} = \cos \omega_{es} T \ \ . \tag{30}$$

The ripple estimation δ_e is given by (24). We deal with the abridging of the impulse response of the maximally flat FIR filters leading to the maximum ripple filters only. This requirement reduces substantially the number of available abridged filters. The standard abridging leads to a feasible FIR filter with more general distribution of zeroes of $Q'(w)$. For the maximum ripple filters the derivation of the pseudoamplitude $Q'(w)$ exhibits zeros in $-1 \leq w \leq 1$ only. The abridging process was investigated for flatness orders $2 \leq p, q \leq 45$.

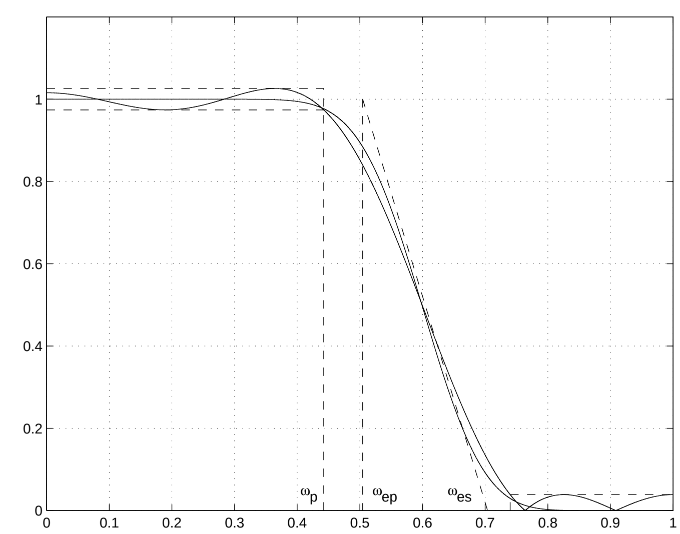

Fig. 2. Amplitude frequency responses $|H(e^{j\omega T})|$ and $|H_a(e^{j\omega T})|$.

The ratio of the lengths of the maximally flat filter and the first abridged maximum ripple filter $(2M+1)/(2L+1)$ varies between 1.7 and 11.

Due to the abridging process the deviation in the passband and stopband is not symmetrically distributed along the unity and zero level, respectively. In order to achieve symmetrical deviation $1 \pm \delta_p$ in the passband and $\pm\delta_s$ in the stopband, an additional normalization of the impulse response is necessary. Frequently the parameters of the abridged filters are very closed to the optimal equiripple filters.

Example 1.
Consider the design of a maximally flat FIR filter with the order of flatness $p = 10$ in the passband and $q = 5$ in the stopband. The length of the filter is $N = 33$. The estimated edges calculated using (30) are $\omega_{ep}T = 0.5046\pi$ and $\omega_{es}T = 0.7039\pi$ and estimation of the ripple limit (24) gives $\delta_e = 0.0613$. The impulse response of the filter $h(n)$ is given in the Tab. II. The amplitude response $|H(e^{j\omega T})|$ is shown in Fig. 2. The impulse response $h_a(n)$ is obtained

TABLE II

THE IMPULSE RESPONSE COEFFICIENTS $h(n)$ - EXAMPLE 1.

n		$h(n)$
0	32	0.000000
1	31	-0.000007
2	30	0.000027
3	29	-0.000009
4	28	-0.000214
5	27	0.000527
6	26	0.000235
7	25	-0.002856
8	24	0.003062
9	23	0.006313
10	22	-0.017165
11	21	0.000246
12	20	0.057800
13	19	0.046946
14	18	-0.081833
15	17	0.294600
16		0.597881

from the $h(n)$ by abridging to the length $2L + 1 = 9$ and consecutive normalizing (Tab. III). The frequency response $|H_a(e^{j\omega T})|$ (Fig. 2) of the abridged filter becomes rippled. The actual edges are $\omega_pT = 0.4424\pi$ and $\omega_sT = 0.7400\pi$ and ripples $\delta_p = 0.0260$, $\delta_s = 0.0388$. An equivalent optimal equiripple filter was designed with the same specifications $2L+1$, ω_p, ω_s and ratio δ_p/δ_s . The impulse responses $h_a(n)$ and $h_{opt}(n)$ of the abridged filter and the optimal equiripple filter are compared in Tab. III.

TABLE III

COMPARISON OF THE IMPULSE RESPONSE COEFFICIENTS.

n		$h_a(n)$	$h_{opt}(n)$
0	8	0.046642	0.046645
1	7	-0.048497	-0.048496
2	6	-0.081303	-0.081305
3	5	0.292691	0.292696
4		0.596549	0.596547

Example 2.
Consider the design of a maximally flat FIR filter with the order of flatness $p = 43$ and $q = 19$. The length of the filter is $N = 127$. The estimated edges (30) are $\omega_{ep}T = 0.5751\pi$ and $\omega_{es}T = 0.6756\pi$. The ripple limit (24) is $\delta_e = 0.0274$. By abridging of the impulse response to the length $2L+1 = 19$ and consecutive normalizing ($h_a(n)$ in the Tab. IV), the frequency response $|H_a(e^{j\omega T})|$ becomes rippled. The actual edges are $\omega_pT = 0.5452\pi$ and $\omega_sT = 0.7038\pi$

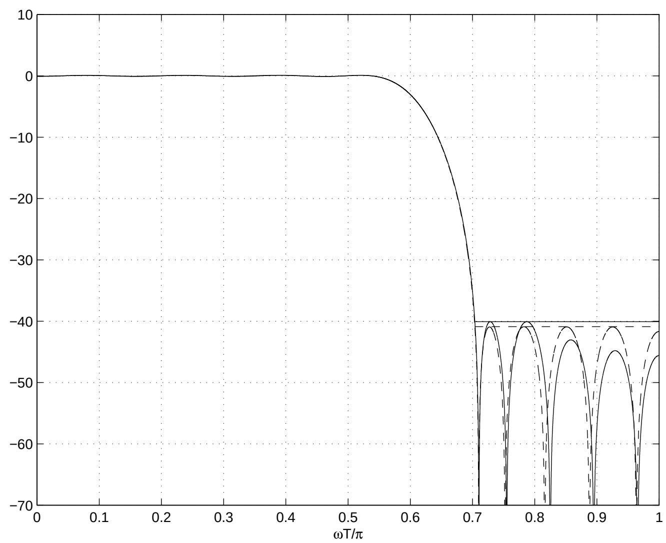

Fig. 3. Comparison of frequency responses 20log$|H_a(e^{j\omega T})|$ and 20log$|H_{opt}(e^{j\omega T})|$.

and ripples $\delta_p = 0.0116$, $\delta_s = 0.0099$. The logarithmic frequency response 20log($|H_a(e^{j\omega T})|$) of the abridged filter is compared in Fig. 3 to the logarithmic frequency response 20log($|H_{opt}(e^{j\omega T})|$) of the optimal equiripple filter with the same specifications $2L + 1$, ω_p, ω_s and ratio δ_p/δ_s.

TABLE IV

THE IMPULSE RESPONSE COEFFICIENTS $h_a(n)$ - EXAMPLE 2.

n		$h_a(n)$	$h_{opt}(n)$
0	24	-0.008575	-0.010018
1	23	0.004643	0.004415
2	22	0.009936	0.009484
3	21	-0.017589	-0.017461
4	20	0.000785	0.000953
5	19	0.028308	0.028408
6	18	-0.029114	-0.029325
7	17	-0.018917	-0.019107
8	16	0.070311	0.070692
9	15	-0.039266	-0.039268
10	14	-0.108107	-0.108273
11	13	0.292790	0.292453
12		0.624213	0.623603

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