Fast Analytical Design Algorithms for FIR Notch Filters

Pavel Zahradník and Miroslav Vlček

Abstract—Fast analytical design procedures for finite impulse response (FIR) maximally flat (MF) and optimal equiripple (ER) notch filters are introduced. The closed form solution provides recursive computation of the impulse response coefficients of the filter. The ER FIR filters are optimal in the Chebyshev sense. The relation between the MF and ER notch filter is presented in order to emphasize the superior performance of the ER narrowband filters over their MF counterparts. The discrete nature of the notch frequency in both filter types is emphasized. Four design examples are included to demonstrate the efficiency of the presented approach.

Index Terms—Analytical design, finite impulse response (FIR), maximally flat (MF) filter, narrow-band filter, notch filter, optimal equiripple (ER) filter.

I. INTRODUCTION

NARROW BAND digital filters are widely used in digital signal processing (DSP). While narrow bandpass filters find their application in the detection of signals the narrow bandstop filters are frequently used in order to remove a single frequency component from the signal spectrum. The narrow bandstop filters are usually called notch filters. In our paper we primarily deal with notch filters but we keep in mind the close relation between these two types of narrow band filters. In our approach to the finite impulse response (FIR) notch filter, we also assume the design of its narrow bandstop counterpart. The design of digital IIR notch filters is rather simple and these filters are frequently used in spite of these filters possess infinite impulse and step responses which can produce spurious signal components that are unwanted in various applications. The IIR notch filters consist of an abridged all-pass second-order section that allows independent tuning of the notch frequency $\omega_n$ and the 3-dB attenuation bandwidth [8].

The main drawback usually emphasized in connection with FIR filters is the higher number of coefficients compared to their IIR counterparts. However, this argument is weakened continuously due to the tremendous advance in DSP and field-programmable-gate-array (FPGA) technology. The number of filter coefficients is no more the most important criterion in the digital filtering. The decisive advantages of FIR filters is the constant group delay and superior time response [15]. Thus, the implementation of FIR filters with 100 coefficients has an practical impact in numerous applications.

A few analytical procedures for the design of linear phase FIR notch filters have recently become available [10]. The methods which lead to feasible filters are generally derived by iterative approximation techniques or by non-iterative, but still numerical procedures, e.g. the window technique. It is worth of noting, that the standard Parks-McClellan algorithm [5] provides rather disappointing results in the design of the FIR notch filters. "The Parks-McClellan iterative design . . . has some inherent limitations. As a result, the numerical solution may fail, especially in the transition region and for notch filters particular" [6]. In this paper, we are concerned with completely analytical design of FIR notch filter of two types, the maximally flat (MF) and the equiripple (ER) ones. For both filter types we introduce the degree formulas which relate the degree of the generating polynomial, the length of the filter, the notch frequency, the width of the notch band and the attenuation in the passbands. We derive the differential equations for the generating polynomials of both filter types. Based on the expansion of the generating polynomials into the Chebyshev polynomials, the recurrent formulae for the direct computation of the impulse response coefficients of both filter types are derived. Consequently, the fast Fourier transform (FFT) algorithm required in the analytical design of optimal narrow band FIR filters [3] is avoided. Both design procedures are recursive and they do not require any FFT algorithm or any iterative technique.

II. POLYNOMIAL APPROXIMATION, ZERO PHASE TRANSFER FUNCTION

Here and in the following we use the independent transformed variable $w$ [11] related to the digital domain by

$$ w = \frac{1}{2} \left( z + z^{-1} \right) \bigg|_{z = e^{j\omega T}} = \cos \omega T $$

confined to the real intervals $(-1, w_s) \cup (w_p, 1)$ We denote $H(z)$ the transfer function of a FIR notch filter with the impulse response $h(m)$ of the length $N$ as

$$ H(z) = \sum_{m=0}^{N-1} h(m) z^{-m}. $$

Assuming an odd length $N = 2n + 1$ and even symmetry of the impulse response

$$ a(0) = h(n), \quad a(m) = 2h(n \pm m), \quad m = 1 ... n $$

Manuscript received December 20, 2002; revised June 6, 2003. This paper was recommended by Associate Editor W.-P. Zhu.

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Digital Object Identifier 10.1109/TCSI.2003.822404
we can write the transfer function of the FIR notch filter
\[ H(z) = z^{-n} \left[ a(0) + \sum_{m=1}^{n} a(m) T_m(w) \right]. \]
(4)

where \( T_m(w) \) is Chebyshev polynomial of the first kind. The frequency response of the filter \( H(e^{j\omega T}) \) can be expressed by the zero phase transfer function \( Q(w) \)
\[ H(e^{j\omega T}) = e^{-j\omega T} Q(\cos \omega T) = z^{-n} Q(w) \bigg|_{z=e^{j\omega T}}. \]
(5)

For \( w = \frac{1}{2} \left( z + z^{-1} \right) \bigg|_{z=e^{j\omega T}} = \cos \omega T \) the zero-phase transfer function \( Q(w) \) represents a polynomial of the real variable \( w \). It reduces to a real valued frequency response of the zero-phase FIR filter. The zero phase transfer function \( Q(w) \) is a polynomial of the real variable \( w \). The zero phase transfer function \( Q(w) \) of the FIR narrow bandpass filter is formed by the generating polynomial \( N_{p,q}(w) \) while the zero phase transfer function \( Q(w) \) of the FIR notch filter is
\[ Q(w) = 1 - N_{p,q}(w). \]
(6)

In our paper we assume two types of the generating polynomials \( N_{p,q}(w) \). We will denote \( A_{p,q}(w) \) the generating polynomial of the FIR MF narrow bandpass filter. The generating polynomial of the FIR ER narrow bandpass filter will be purposely denoted as \( Z_{p,q}(w) \).

III. FIR Maximally Flat Notch Filter

For the design of FIR MF notch filter we propose the generating polynomial \( A_{p,q}(w) \) of the FIR MF narrow bandpass filter introduced in [12]
\[ N_{p,q}(w) = A_{p,q}(w) = C(1-w)^p(1+w)^q. \]
(7)

The notation \( A_{p,q}(w) \) emphasizes that \( p \) counts multiplicity of zeros at \( w = 1 \) and \( q \) corresponds to multiplicity of zeros at \( w = -1 \). Forming of the derivative of the polynomial
\[ \frac{dA_{p,q}(w)}{dw} = \]
\[ -Cp(1-w)^{p-1}(1+w)^q + Cq(1-w)^p(1+w)^{q-1} \]
and by simple manipulation of (7)
\[ (1-w)(1+w)\frac{dA_{p,q}(w)}{dw} = \]
\[ = -p(1+w)A_{p,q}(w) + q(1-w)A_{p,q}(w) \]
we arrive to the differential equation for the generating polynomial \( A_{p,q}(w) \)
\[ (1-w^2)\frac{dA_{p,q}(w)}{dw} + [p - q + (p + q)w] A_{p,q}(w) = 0. \]
(10)
The differential equation (10) for the polynomial \( A_{p,q}(w) \) forms a completely new concept in digital filter design as it provides the recursive evaluation of the impulse response coefficients of the filter described in Section V. The normalization of the generating polynomial \( A_{p,q}(w) \) constrains \( A_{p,q}(w_m) = 1 \) where \( w_m \) is the position of the maximum of the generating polynomial \( A_{p,q}(w) \) as illustrated in Fig. 1. The normalization of the generating polynomial \( A_{p,q}(w) \) results in
\[ A_{p,q}(w) = \left[ \frac{p+q}{2p} (1-w) \right]^p \left[ \frac{p+q}{2q} (1+w) \right]^q. \]
(11)

The polynomial
\[ Q_A(w) = 1 - A_{p,q}(w) \]
(12)
\[ = 1 - \left[ \frac{p+q}{2p} (1-w) \right]^p \left[ \frac{p+q}{2q} (1+w) \right]^q \]
represents the real-valued zero phase transfer function of the FIR MF notch filter of the real variable \( w = \cos \omega T \). For

Fig. 1. Generating polynomial \( A_{3,37}(w) \) (dashed) and the zero phase transfer function \( Q_A(w) = 1 - A_{3,37}(w) \) of the FIR MF notch filter with extremal value for \( w_m = (37 - 3)/(37 + 3) = 0.85 \).

Fig. 2. Amplitude frequency response \( |H(e^{j\omega T})| \) based on the generating polynomial \( Q_A(w) = 1 - A_{3,37}(w) \) from Fig. 1 with minimum value for \( \omega_m T = \arccos((37 - 3)/(37 + 3)) = 0.1766 \pi \).
illustration, the zero phase transfer function of the FIR MF notch filter \( Q_A(w) = 1 - A_{3,37}(w) \) is shown in Fig. 1. The corresponding amplitude frequency responses \( |H(e^{j\omega T})| \) and \( 20 \log |H(e^{j\omega T})| \) [dB] are shown in Fig. 2 and Fig. 3. The transfer function of the FIR MF notch filter is

\[
H(z) = \sum_{m=0}^{N-1} h(m) = z^{-m} \left( 1 - A(p,q)(w) \right).
\]

IV. NOTCH FREQUENCY AND THE DEGREE OF THE MAXIMALLY FLAT NOTCH FILTER

The notch frequency \( \omega_m T \) is derived from the minimum value of the zero phase transfer function \( Q_A(w) \) (12)

\[
(1 - w^2) \frac{d}{dw} Q_A(w) = - \left[ \frac{p + q}{2p} (1 - w) \right]^p \left[ \frac{p + q}{2q} (1 + w) \right]^q \times [q - p - (p + q)w] = 0,
\]

as

\[
\omega_m = \cos \omega_m T = \frac{q - p}{q + p},
\]

we obtain a symmetrical FIR MF notch filter based on the symmetrical generating polynomial \( A_{p,q}(w) \) centered around \( \omega_m = 0 \) in the \( w \)-domain (Fig. 4), and around \( \omega_m T = \pi/2 \) in the frequency domain \( \omega \) (Fig. 5). In the \( w \)-domain the width of the notchband \( \Delta w \) of the symmetrical FIR MF notch filter (16) can be for the specified attenuation in the passbands \( a \) [dB] (Fig. 4) derived from equation (12). For (16) and \( Q_A(w) = 10^{0.05a} \) we obtain

\[
\Delta w = 2 \sqrt{1 - (1 - 10^{-0.05a})^{2/n}}, \quad n = \frac{2 \log (1 - 10^{-0.05a})}{\log \left( 1 - \frac{\Delta w}{2} \right)}. \tag{17}
\]

In the frequency domain \( \omega \) (Fig. 5) the equivalent formulas

\[
\Delta \omega T = \pi - 2 \arccos \sqrt{1 - (1 - 10^{-0.05a})^{2/n}} \tag{18}
\]

and

\[
n = \frac{\log (1 - 10^{-0.05a})}{\log \cos \frac{\Delta \omega T}{2}} \tag{19}
\]

are available. We call (19) the degree equation of the FIR MF notch filter. The calculated degree \( n \) (19) is, in principle, real number, so up-rounding to integer value

\[
n = \left[ n + \frac{1}{2} \right] \tag{20}
\]

is necessary. The brackets \([\cdot]\) in (20) denote the rounding operation. For the symmetrical FIR MF notch filter and the standard attenuation in the passbands \( a = 20 \log(\sqrt{2}/2) = -3.0103 \) dB the relation between the width of the notchband \( \Delta \omega T \) and the filter length \( N = 2n + 1 \) is depicted in Fig. 6. For the non-symmetrical FIR MF filter specified by \( p \neq q \) of the length \( N = 2(p + q) + 1 \) the width of the notchband \( \Delta \omega T \) depends on the position of the notch frequency \( \omega_m T \) as demonstrated in Fig. 7. It is apparent that the "worst case" i.e. the widest notchband occurs for the symmetrical notch filter (16) for which the filter length \( N = 2n + 1 \) is exactly expressed (19). When moving the notch frequency \( \omega_m T \) along the frequency axis to the edges of the frequency interval \( \omega_m T \rightarrow 0 \) or \( \omega_m T \rightarrow \pi \), the width of the notchband \( \Delta \omega T \) monotonically decreases by a few percent only, as quantified in Table I. For reasonable filter lengths the width of the

<table>
<thead>
<tr>
<th>TABLE I</th>
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<tbody>
<tr>
<td>( N )</td>
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<td>181</td>
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<td>201</td>
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</tbody>
</table>
passbands \( a \) [dB] for any FIR MF notch filter with excellent accuracy. The relations for the integer values \( p, q \)

\[
p = \left\lfloor n \sin^2 \left( \frac{\omega_m T}{2} \right) \right\rfloor, \quad q = \left\lfloor n \cos^2 \left( \frac{\omega_m T}{2} \right) \right\rfloor
\]

(21)

follow from (15) and from the degree \( n = p + q \) (20). The brackets \( \lfloor \cdot \rfloor \) in (21) denote again the rounding operation.

V. IMPULSE RESPONSE COEFFICIENTS OF THE FIR MAXIMALLY FLAT FILTER

We can express the generating polynomial \( A_{p,q}(w) \) of the degree \( n = p + q \) as the sum of Chebyshev polynomials of the first kind \( T_m(w) \)

\[
A_{p,q}(w) = \sum_{m=0}^{n} a(m) T_m(w).
\]

(22)

The coefficients \( a(m) \) define the impulse response \( h(m) \) (3) of the length \( N = 2(p + q) + 1 \). Assuming the generating polynomial \( A_{p,q}(w) \) of the FIR MF narrow bandpass filter in the sum (22) we can write

\[
(1 - w^2) \frac{dA_{p,q}(w)}{dw} = \sum_{m=1}^{n} a(m)(1 - w^2) \frac{dT_m(w)}{dw} =
\]

(23)
TABLE II
Recursive algorithm for evaluation of the coefficients $a(m)$ of the generating polynomial $A_{p,q}(w)$. 

<table>
<thead>
<tr>
<th>$p$, $q$</th>
<th>given</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = p + q$</td>
<td>$a(n + 1) = 0$</td>
</tr>
<tr>
<td>$a(n) = (-1)^p 2^{-(p-q+1)} \left( \frac{p+q}{2p} \right)^p \left( \frac{p+q}{2q} \right)^q$</td>
<td>$T_{m-1}(w) - T_{m+1}(w)$</td>
</tr>
</tbody>
</table>

**Body** (for $k = n + 1$ to $3$)

$$a(k - 2) = \frac{(n + k)a(k) + 2(2p - n)a(k - 1)}{n + 2 - k}$$

$$(end\ loop\ on\ k)$$

$$a(0) = \frac{(n + 2)a(2) + 2(2p - n)a(1)}{2n}$$

By introducing (22) and (23) into the differential equation (10) and using the recursive formula for Chebyshev polynomials

$$T_{m+1}(w) = 2wT_m(w) - T_{m-1}(w) \quad (24)$$

we get the identity

$$\sum_{m=1}^{n} a(m) \frac{m}{2} [T_{m-1}(w) - T_{m+1}(w)] + (p - q)a(0) + \sum_{m=1}^{n} a(m)(p-q)T_m(w) + (p+q)a(0)w + \sum_{m=1}^{n} a(m)(p+q)\frac{1}{2} [T_{m-1}(w) + T_{m+1}(w)] = 0 \quad (25)$$

By iterating eq. (25) we have deduced a simple recursive algorithm for the evaluation of the coefficients $a(m)$ of the generating polynomial $A_{p,q}(w)$ of the FIR MF narrow bandpass filter. The recursive algorithm is presented in Table II. The coefficients $h(m)$ of the impulse response of the FIR MF notch filter are obtained from the coefficients $a(m)$ of the FIR MF narrow bandpass filter as follows

$$h(n) = 1 - a(0), \quad h(n \pm m) = -\frac{a(m)}{2}, \quad m = 1 \ldots n \quad (26)$$

VI. DESIGN OF FIR MAXIMALLY FLAT NOTCH FILTER

The goal of the FIR MF notch filter design is to find the two integer values $p$ and $q$ in order to satisfy the filter specification as precisely as possible. The design procedure is as follows:

1) Specify the notch frequency $\omega_n T$, maximal width of the notchband $\Delta \omega T$ and the attenuation in the passbands $a$ [dB] as demonstrated in Fig. 3.

2) Calculate the minimum degree $n$ (19), (20) required to satisfy the filter specification.

3) Calculate the integer values $p$ and $q$ (21).

4) Check the notch frequency (15) for obtained $p$, $q$.

5) Evaluate the coefficients $a(m)$ of the generating polynomial $A_{p,q}(w)$ recursively (Tab. II).

6) Evaluate the coefficients of the impulse response $h(m)$ of the FIR MF notch filter (26).

It is worthy of note that a substantial part of coefficients of the impulse response $h(m)$ of the FIR MF notch filter has negligible values. From this fact follows the possible large abbreviation of the impulse response of the FIR MF notch filter by the rectangular windowing without significant deterioration of the frequency properties of the filter as emphasized in [12].

VII. EXAMPLES OF THE DESIGN OF FIR MAXIMALLY FLAT NOTCH FILTER

**Example No. 1 - Design the FIR MF notch filter specified by $\omega_n T = 0.35 \pi$ and $\Delta \omega T = 0.15 \pi$ for $a = -3.0103$ dB.**

Using our design procedure (Section VI) we get $n = [43.8256] \rightarrow 44$ (19), (20), $p = [11.9644] \rightarrow 12$ and $q = [31.8610] \rightarrow 32$ (21). The filter length is $N = 89$ coefficients. The actual filter parameters are $\omega_n T = 0.3498 \pi$
and $\Delta \omega T = 0.1496 \pi$. The coefficients $a(m)$ were evaluated recursively (Table II). The coefficients of the impulse response $h(m)$ of the FIR MF notch filter were evaluated by (26). Because $|h(m)| < 10^{-6}$ for $0 \leq m \leq 13$ and $75 \leq m \leq 88$, only the 71 central coefficients of the impulse response $h(m)$ for $14 \leq m \leq 74$ are summarized in Table III. The amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB] of the FIR MF notch filter is shown in Fig. 8.

**Example No. 2 - Let us design the FIR MF notch filter specified by $\omega_m T = 0.84 \pi$ and $\Delta \omega T = 0.122 \pi$ for $a = -3.0103$ dB.**

Using our design procedure (Section VI) we get $n = \lceil 66.462 \rceil \rightarrow 67$ (19), $p = \lceil 62.8563 \rceil \rightarrow 63$ and $q = \lceil 4.1437 \rceil \rightarrow 4$ (21). The filter length is $N = 135$. The actual filter parameters are $\omega_m T = 0.8428 \pi$ and $\Delta \omega T = 0.1206 \pi$. The coefficients $a(m)$ were evaluated recursively (Table II). The coefficients of the impulse response $h(m)$ of the notch filter evaluated by (26) are summarized in Table IV. Because $|h(m)| < 10^{-6}$ for $0 \leq m \leq 30$ and $104 \leq m \leq 134$, only the 73 central coefficients of the impulse response $b(m)$ for $31 \leq m \leq 103$ are summarized in Tab. IV. The amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB] of the FIR MF notch filter is shown in Fig. 9.

### VIII. FIR EQUIRIPPLE NOTCH FILTER

The FIR equiripple notch filter is based on the Zolotarev polynomial. The Zolotarev polynomial $Z_{p,q}(u|k)$ results from the Zolotarev’s first approximation problem [2], requiring a polynomial which deviates at least from zero (in equiripple Chebyshev sense) in the two disjoint intervals as shown in Fig. 10. Zolotarev derived the general solution of his approximation problem in terms of Jacobi’s elliptic functions

$$Z_{p,q}(u|k) = \frac{(-1)^p}{2} \left[ \frac{H(u - \frac{p}{n} K(\kappa))}{H(u + \frac{p}{n} K(\kappa))} \right]^n$$

where $H \left( u \pm \frac{p}{n} K(\kappa) \right)$ is the Jacobi’s Eta function, $\text{sn}(u|\kappa)$, $\text{cn}(u|\kappa)$, $\text{dn}(u|\kappa)$ are Jacobi’s elliptic functions, $K(\kappa)$ is the quarter-period given by the complete elliptic integral of the first kind of the Jacobi’s elliptic modulus $\kappa$. The degree of the Zolotarev polynomial is $n = p + q$.

A comprehensive treatise of the Zolotarev polynomials including the analytical solution of the coefficients of Zolotarev polynomials with the algebraic evaluation of the Jacobi’s Zeta function $Z(\frac{p}{n} K(\kappa)|\kappa)$ and the elliptic integral of the third kind $\Pi(\sigma_m \frac{p}{n} K(\kappa)|\kappa)$ of the discrete argument was presented in [13]. In the following, we briefly recall elementary facts useful in the design of FIR ER notch filters. Assuming the conformal

### Table IV

<table>
<thead>
<tr>
<th>m</th>
<th>$h(m)$</th>
<th>m</th>
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<td>31</td>
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<td>48</td>
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<td>67</td>
<td>0.903713</td>
</tr>
</tbody>
</table>

### Fig. 8

Amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB] based on the zero phase transfer function $Q(w) = 1 - A_{12,32}(w)$, see example No. 1.

### Fig. 9

Amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB] based on the zero phase transfer function $Q(w) = 1 - A_{13,43}(w)$, see example No. 2.
where we denote $Z_{p,q}(w) = Z_{p,q}(u|\kappa)$ the Zolotarev polynomial in the $w$-domain. The notation $Z_{p,q}(w)$ for Zolotarev polynomial emphasizes that $p$ counts the number of zeros right from the maximum $w_m$ and $q$ corresponds to the number of zeros left from the maximum $w_m$. The extremal values of Zolotarev polynomial $Z_{p,q}(w)$ alternates between $-1$ and $+1$ ($q+1$)-times in the interval $(-1, w_s)$ and $(p+1)$-times in the interval $(w_p, 1)$ as shown in Fig. 10. It was also shown in [13] that the zero phase transfer function $Q_Z(w)$ of the FIR ER notch filter derived from the Zolotarev polynomial

$$Q_Z(w) = 1 - \frac{Z_{p,q}(w)}{y_m + 1}.$$

For illustration the zero phase transfer function $Q_Z(w)$ of

$$Q_Z(w) = 1 - \frac{Z_{p,q}(w)}{y_m + 1}.$$

The position of the maximum value $y_m = Z_{p,q}(w_m)$ is

$$w_m = w_s + 2 \frac{\operatorname{sn} \left( \frac{p}{n} K(\kappa) | \kappa \right) \operatorname{cn} \left( \frac{p}{n} K(\kappa) | \kappa \right)}{\operatorname{dn} \left( \frac{p}{n} K(\kappa) | \kappa \right)} Z \left( \frac{p}{n} K(\kappa) | \kappa \right).$$

(32)
Chebyshev polynomials of the first kind $T_m(w)$

$$Z_{p,q}(w) = \sum_{m=0}^{n} a(m) T_m(w)$$  \hspace{1cm} (41)

as derived and presented in [13]. The algorithm for the
evaluation of the coefficients $a(m)$ is summarized in Table V.
The impulse response coefficients $h(m)$ of the FIR ER notch
filter are obtained by the normalization of the coefficients
$a(m)$ as follows

$$h(n) = \frac{y_m - a(0)}{y_m + 1}, \quad h(n \pm m) = -\frac{a(m)}{2(y_m + 1)}, \quad m = 1 \ldots n.$$  \hspace{1cm} (42)

The transfer function of the FIR ER notch filter is

$$H(z) = \sum_{m=0}^{N-1} h(m) z^{-m} = z^{-n} \left(1 - \frac{Z_{p,q}(w) + 1}{y_m + 1} \right).$$  \hspace{1cm} (43)

The FIR optimal ER narrow band filters based on Zolotarev
polynomials are the maximum ripple filters. The only available
band edges are discretized by the partition equation (33). Consequen-
tly, the width of the notchband $\Delta \omega T = \omega_s T - \omega_p T$
is discretized, too. For details regarding the discrete nature of
the notch frequency $\omega_m T$ see Sections XI and XII. This
is naturally different from the filters designed by the Parks-
McClellan program where the band edges are adjusted by
one or more extra zeros which are off the unit circle [5].
The Zolotarev polynomials have no other zeros than those on
the unit circle and therefore they satisfy only the band-edge
requirements constrained by the partition equation (33). Strict
approximation requirements usually give such discrete values
for the positions of the band edges.

**IX. DESIGN OF FIR EQUIRIPPLE NOTCH FILTER**

The goal of the FIR ER notch filter design is to find the
three parameters $p$, $q$ and $\kappa$ in order to satisfy the specified
notch frequency $\omega_m T$, width of the notchband $\Delta \omega T$ and the
attenuation in the passbands $a$ [dB], as precisely as possible.
Our design procedure for the optimal FIR notch filters is
a significantly simplified version of that given by X. Chen
and T. W. Parks [3]. In contrast to [3], our procedure is a
completely analytical one, it is free of the transformation from
$(-1,1) \cup (\alpha, \beta)$ to the digital $w$-domain $(-1, w_s) \cup (w_p, 1)$
and it does not require any FFT algorithm. The design procedure
is as follows:

1) Specify the notch frequency $\omega_m T$, width of the notch-
band $\Delta \omega T$ and the attenuation in the passbands $a$ [dB]
as demonstrated in Fig. 13.

2) Calculate the band edges

$$\omega_p T = \omega_m T - \frac{\Delta \omega T}{2}, \quad \omega_s T = \omega_m T + \frac{\Delta \omega T}{2}.$$  \hspace{1cm} (44)

The relations (44) holds for $k < 0.95$ with excellent
accuracy, see Section XI.

3) Evaluate the Jacobi’s elliptic modulus $\kappa$

$$\kappa = \sqrt{1 - \frac{1}{\tan^2(\varphi_s) \tan^2(\varphi_p)}}.$$  \hspace{1cm} (45)

from Fig. 10 is shown in Fig. 11. The relations between the
attenuation in the passbands $a$ [dB] of the FIR ER notch
filter and the maximum value $y_m$ of the Zolotarev polynomial are

$$a = 20 \log \left(1 - \frac{2}{y_m + 1}\right), \quad y_m = \frac{2}{1 - 10^{0.05 a}} - 1.$$  \hspace{1cm} (39)

The ripples $\delta$ (Fig. 11) of the FIR ER notch filter are related
to the maximum value $y_m$ by the formula

$$\delta = \frac{1}{y_m + 1}.$$  \hspace{1cm} (40)

The algebraic solution of the linear differential equation (29)
provides expansion of the Zolotarev polynomial in terms of

Fig. 12. Amplitude frequency response $|H(e^{j\omega T})|$ based on the zero phase
transfer function $Q_Z(w)$ from Fig. 11. The parameters are $\omega_p T = 0.3506 \pi$, $\omega_m T = 0.4006 \pi$, $\omega_s T = 0.507 \pi$, $\Delta \omega T = 0.1001 \pi$ and $10^{0.05 a} = 1 - 2/(y_m + 1) = 0.6868$.

Fig. 13. Amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB] based on
the zero phase transfer function $Q_Z(w)$ from Fig. 11. The parameters are $\omega_p T = 0.3506 \pi$, $\omega_m T = 0.4006 \pi$, $\omega_s T = 0.4507 \pi$, $\Delta \omega T = 0.1001 \pi$ and $a = -3.2634$ dB.
for the auxiliary parameters $\varphi_s$ and $\varphi_p$

$$\varphi_s = \frac{\omega_s T}{2} , \quad \varphi_p = \frac{\pi - \omega_p T}{2} .$$

(46)

4) Calculate the rational values $p/n$ and $q/n$ using (34).
5) Determine the required maximum value $\nu_m$ (39).
6) Calculate the minimum degree $n$ required to satisfy the filter specification using the degree equation (35). Round up the degree $n$ (20). The calculation of the degree $n$ by

$$\text{the degree equation (35) requires the evaluation of the Jacobi's Zeta function } \Pi(\sigma_m, \frac{p}{n} K(\kappa) | \kappa) \text{ and the evaluation of the elliptic integral of the third kind } \Pi(\sigma_m, \frac{p}{n} K(\kappa) | \kappa) \text{ both of the continuous argument, as the particular integer values } p, q \text{ and } n \text{ are still not known in this step. From (34) the rational values } p/n \text{ and } q/n \text{ are known only. We propose to evaluate both functions algebraically. For details see Appendix.}$

7) Calculate the integer values $p$ and $q$ defining the Zolotarev polynomial $Z_{p,q}(w)$

$$p = \left\lfloor n \frac{F(\varphi_s | \kappa)}{K(\kappa)} \right\rfloor , \quad q = \left\lfloor n \frac{F(\varphi_p | \kappa)}{K(\kappa)} \right\rfloor .$$

(47)

The brackets $\lfloor \rfloor$ in (47) stand for the rounding.

8) Calculate the actual band edges $\omega_p T, \omega_s T$

$$\omega_p T = \arccos(w_p) , \quad \omega_s T = \arccos(w_s) \quad (48)$$

for (30), (31) and the notch frequency $\omega_m$.

$$\omega_m T = \arccos(w_m) .$$

(49)

for (32). The evaluation of (32) requires the evaluation of the Jacobi's Zeta function $Z(\frac{p}{n} K(\kappa) | \kappa)$ of the discrete argument. For details see Appendix.
9) Calculate the actual attenuation in the passbands $a$ [dB] (39) for the corresponding maximum $y_m$ (37). The calculation of the maximum value $y_m$ in (37) requires the evaluation of the Jacobi’s Zeta function $Z(m, \frac{\pi}{2})(\kappa)$ and the evaluation of the elliptic integral of the third kind $\Pi(s_m, m, \frac{\pi}{2})(\kappa)$ both of the discrete argument, as the particular integer values $p, q$ are known from step 7. We propose to evaluate both functions algebraically. For details see Appendix.

10) For the integer values $p, q$ and the elliptic modulus $\kappa$ evaluate the coefficients $a(m)$ (41) of the Zolotarev polynomial $Z_{p,q}(\omega)$ (Tab. V).

11) Calculate the impulse response coefficients $h(m)$ of the FIR ER notch filter (42).

X. EXAMPLES OF THE DESIGN OF FIR EQUIRIPPLE NOTCH FILTER

Example No. 3 - Design the FIR ER notch filter specified by $\omega_m T = 0.35 \pi$ (the same value as specified in example No. 1) and $\Delta \omega T = 0.075 \pi$ (half of the value specified in example No. 1) for $a = -1$ dB.

Using our design procedure (Section IX) we obtain $\omega_p = 0.3125 \pi$, $\omega_s = 0.3875 \pi$ (44), $\varphi_s = 0.6087$, $\varphi_p = 1.0799$ (46), $\kappa = 0.641747$ (45), $n = \lfloor 30.0562 \rfloor = 31$ (35), $p = 10.8323 \rightarrow 11$, $q = 20.1677 \rightarrow 20$ (47). For the calculated values $p, q, \kappa$ the actual filter parameters are $\omega_m T = 0.3554 \pi$ (49), $\Delta \omega T = 0.0756 \pi$ (48) and $a = -0.8648$ dB (39). The filter length is $N = 63$ coefficients. The coefficients $a(m)$ were evaluated recursively (Table V). The coefficients of the impulse response $h(m)$ of the FIR ER notch filter evaluated by (42) are summarized in Table VI. The amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB] of the designed FIR ER notch filter is shown in Fig. 14.

![Fig. 14. Amplitude frequency response $20 \log |H(e^{j\omega T})|$ based on the Zolotarev polynomial $Z_{11,20}(\omega)$, $\kappa = 0.641747$, $N = 63$, see example No. 3.](image)

| m | $h(m)$ | $|h(m)|$ |
|---|---|---|
| 0 | 0.029503 | 0.015915 |
| 1 | 0.005671 | 0.032502 |
| 2 | -0.007359 | 0.014297 |
| 3 | -0.013952 | 0.023465 |
| 4 | -0.004552 | 0.034402 |
| 5 | 0.012180 | 0.006201 |
| 6 | 0.016879 | 0.030588 |
| 7 | 0.001786 | 0.033901 |
| 8 | -0.017745 | 0.003115 |
| 9 | -0.018643 | 0.036435 |
| 10 | 0.002714 | 0.030855 |
| 11 | 0.023472 | 0.009956 |
| 12 | 0.018694 | 0.040274 |
| 13 | -0.008769 | 0.025461 |
| 14 | -0.028640 | 0.018239 |
| 15 | -0.016658 | 0.011009 |

Example No. 4 - Design the FIR ER notch filter specified by $\omega_m T = 0.84 \pi$ (the same value as specified in example No. 2) and $\Delta \omega T = 0.061 \pi$ (half of the value specified in example No. 2) for $a = -0.95$ dB.

Using our design procedure (Section IX) we obtain $\omega_p = 0.8095 \pi$, $\omega_s = 0.8705 \pi$ (44), $\varphi_s = 0.2992$, $\varphi_p = 1.3674$ (46), $\kappa = 0.743509$ (45), $n = \lfloor 37.2896 \rfloor = 38$ (35), $p = 31.9713 \rightarrow 32$, $q = [6.0287] \rightarrow 6$ (47). For the calculated values $p, q, \kappa$ the actual filter parameters are $\omega_m T = 0.8408 \pi$ (49), $\Delta \omega T = 0.0607 \pi$ (48) and $a = -0.9109$ dB (39). The filter length is $N = 77$ coefficients. The coefficients $a(m)$ were evaluated recursively (Table V). The coefficients of the impulse response $h(m)$ of the FIR ER notch filter evaluated by (42) are summarized in Table VII. The amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB] of the designed FIR ER notch filter is shown in Fig. 15.

| m | $h(m)$ | $|h(m)|$ |
|---|---|---|
| 0 | 0.025967 | 0.023583 |
| 1 | 0.009026 | 0.016098 |
| 2 | 0.001691 | 0.004079 |
| 3 | 0.002540 | 0.006927 |
| 4 | 0.003215 | 0.015185 |
| 5 | 0.008960 | 0.026644 |
| 6 | 0.013104 | 0.027872 |
| 7 | 0.014293 | 0.021678 |
| 8 | 0.011819 | 0.008872 |
| 9 | 0.005907 | 0.006575 |
| 10 | 0.002247 | 0.020828 |
| 11 | 0.010702 | 0.030245 |
| 12 | -0.017226 | 0.023232 |
| 13 | -0.029891 | 0.026411 |
| 14 | -0.017624 | 0.013833 |
| 15 | 0.001053 | 0.002336 |
| 16 | -0.000216 | 0.018080 |
| 17 | -0.011026 | 0.029453 |
| 18 | 0.020271 | 0.091626 |
| 19 | -0.024966 | 0.091626 |
XI. RELATION OF FIR MAXIMALLY FLAT AND FIR EQUIRipple NOTCH FILTER

The width of the notchband $\Delta \omega T$ we can obtain in maximally flat design is relatively broad. Even for the impulse response of the length $N \sim 200$ coefficients the width of the notchband $\Delta \omega T$ gives approximately $\pi/10$ and for further increasing filter length $N$ the width of the notchband $\Delta \omega T$ decreases rather slowly as demonstrated in Fig. 6. On the other hand the FIR ER notch filters achieve significantly narrower width of the notchband $\Delta \omega T$ compared to the FIR MF filter of the same length. The price for it are the allowed ripples $\delta$ in the passbands. The performance of the FIR MF and FIR ER notch filter is compared in Fig. 16. For the integer values $p = 61$, $q = 4$ ($N = 131$) the parameters of the FIR MF notch filter are $\omega_m T = 0.8404 \pi$ and $\Delta \omega T = 0.1225 \pi$ for $\alpha = -3.0103$ dB. For the attenuation in the passbands $\alpha = -9.109$ dB the width of the notchband amounts $\Delta \omega T = 0.1664 \pi$. The FIR ER notch filter is identical with that one from Example No.4. The FIR MF notch filter is defined by the two integer values $p$ and $q$ only. Let us recall the relation (15) for the notch frequency of the FIR MF notch filter

$$\omega_m T = \arccos \left( \frac{q - p}{q + p} \right).$$

(50)

The strict discretization of the available notch frequencies $\omega_m T$ as demonstrated in Fig. 7 follows from (50). The FIR ER notch filter is defined by the two integer values $p$, $q$ and one additional parameter which is the elliptic modulus $\kappa$. The elliptic modulus $\kappa$ considerably affects the maximum value $y_m$ and thus the ripples $\delta$ (40) and the attenuation in the passbands $a$ [dB] (39) of the filter. If $\kappa$ increases the ripples $\delta$ decrease. The elliptic modulus $\kappa$ affects the notch frequency of the FIR ER filter in specific manner, too. In order to differentiate between the notch frequency $\omega_m T$ of the FIR MF notch filter and the notch frequency of the FIR ER notch filter, we will denote here, and in the next section, the notch frequency of the FIR ER notch filter $\omega_m(\kappa) T$ to emphasize its dependence on the elliptic modulus $\kappa$. The notch frequency $\omega_m(\kappa) T$ gives in the limit case $\kappa \to 0$ the value

$$\omega_m(0) T = \frac{p}{q + p} \pi$$

(51)

which results from (32), (49). In the opposite limit case $\kappa \to 1$ the notch frequency $\omega_m(\kappa) T$ resulting from (32), (49) gives

$$\omega_m(1) T = \arccos \left( \frac{q - p}{q + p} \right).$$

(52)

The influence of the elliptic modulus $\kappa$ for the FIR ER notch filter based on the Zolotarev polynomial. The plot of the notch frequency $\omega_m(\kappa) T$ as a function of the elliptic modulus $\kappa$ is shown by the solid line in Fig. 17. For the filter specifications occurring in the practical filter design the Jacobi's elliptic modulus $\kappa$ is operated in the range $0.6 < \kappa < 0.85$. An extremely important property of the FIR ER notch filters for our next conclusion is, that for $\kappa < 0.85$ the notch frequency $\omega_m(\kappa) T$ is almost constant equal to the limit value $\omega_m(0) T$, see Fig. 17. Hence in the practical design of the FIR ER notch filter the available notch frequencies $\omega_m(\kappa) T$ are of a discrete nature expressed by (51). However this discrete nature of the notch frequency $\omega_m(\kappa) T$ is "less strict" compared to that of the FIR MF filter (50) due to the very small dependence of the notch frequency $\omega_m(\kappa) T$ on the elliptic modulus $\kappa$. For increasing $\kappa > 0.85$ the notch frequency $\omega_m(\kappa) T$ changes more rapidly. For $k > 0.95$ the ripples $\delta$ become extremely small and it is no more advantageous to design the notch filter as an FIR ER one. Hence for $k > 0.95$ we propose to design the FIR notch filter using the procedure for the FIR MF notch filter described in Section VI. For $\kappa > 0.99$ the notch frequency $\omega_m(\kappa) T$ converges abruptly but continuously to the notch frequency $\omega_m T$ of the FIR MF filter. The notch frequency $\omega_m T$ of the FIR MF filter based

![Fig. 16. Comparison of the amplitude frequency responses 20 log $|H(e^{j}\omega T)|$ [dB] of the FIR MF notch filter of the length $N = 131$ and the FIR ER notch filter of the length $N = 77$](image)
on the polynomial $A_{12,4}(w)$ is denoted by the dotted line in Fig. 17. For $\kappa = 1$ the ripples $\delta$ are equal to zero and the FIR equiripple notch filter becomes maximally flat. The dashed curves in Fig. 17 demonstrate the course of the band edges $\omega_a(\kappa)T$ and $\omega_p(\kappa)T$ of the FIR ER notch filter. The limit values are $\omega_a(0)T = \omega_a(\kappa)T = \omega_a(0)T$, while $\omega_a(1)T = \pi$ and $\omega_p(1)T = 0$. The average frequency

$$\omega_a(\kappa)T = \frac{\omega_a(\kappa)T + \omega_p(\kappa)T}{2}$$

(53)
is denoted by the dash-dotted curve in Fig. 17. An excellent coincidence of the average frequency $\omega_a(\kappa)T$ and the notch frequency $\omega_m(\kappa)T$ for $\kappa < 0.95$ is apparent in Fig. 17. The second step (44) in our design procedure is based on this fact. The limit values of the average frequency are $\omega_a(0)T = \omega_a(0)T$ and $\omega_a(1)T = \pi/2$. In our particular case for $p = 12$, $q = 4$ the notch frequency (50) of the FIR MF notch filter is $\omega_mT = 0.6667\pi$. Using eq. (51) the limit value of the notch frequency of the FIR ER notch filter for $p = 12$, $q = 4$, $\kappa = 0$ is $\omega_m(0)T = 0.75\pi$. From (50) and (52) it follows that the notch frequency of the FIR MF notch filter $\omega_mT$ is equal to the notch frequency of the FIR ER notch filter $\omega_m(\kappa)T$ for the same integer values $p$, $q$ and $\kappa = 1$. Furthermore, from (27) and (37) it follows that

$$\lim_{\kappa \to 1} \left\{ \frac{Z_{p,q}(u)\omega}{y_m} \right\} = \left[ \frac{p + q}{2p} (1 - u) \right]^p \left[ \frac{p + q}{2q} (1 + u) \right]^q = A_{p,q}(\omega) \ .$$

(54)

Finally, we conclude that for the given integer values $p$, $q$ the FIR maximally flat notch filter is the limit case of the FIR optimal ER notch filter with the elliptic modulus $\kappa \to 1$.

![Fig. 17. Course of important frequencies of the FIR equiripple notch filter based on the Zolotarev polynomial $Z_{12,4}(w)$.](image)

XII. DISCRETE NATURE OF THE NOTCH FREQUENCY AND THE DESIGN OF THE FIR NOTCH FILTER

As emphasized in the previous section, both the notch frequency $\omega_mT$ of the FIR MF notch filter and the notch frequency $\omega_m(\kappa)T$ of the FIR ER notch filter are of discrete nature. Due to this fact the main challenge in the analytical design of FIR notch filters is to obtain the proper related integer values $p$, $q$. The effort to achieve the specified notch frequency as precisely as possible may result in lower or higher filter length $N$ compared to the value obtained by the degree formulas (19), (35). Consequently more relaxed or more strict values of the width of the notchband $\Delta \omega T$ and attenuation in the passbands $\alpha$ [dB] may be expected. The insight into the discrete nature of the notch frequency as presented in the previous section simplifies the design of the FIR notch filter with optimally tuned notch frequency with respect to the remaining filter parameters $N$, $\Delta \omega T$ and $\alpha$ [dB]. Now, we will discuss the alternatives to our design examples presented in Sections VII and X. In our examples we have evaluated the actual filter parameters and impulse responses for the lowest acceptable filter length $N$ obtained by the degree formulas (19), (35). The dashed lines in Fig. 18-21 demarcate the allowable discrete values $p$, $q$ for the minimal degree $n = p+q$ based on the degree formulas (19) and (35). The integer values $p$, $q$ resulting from the filter design are marked by the filled circle in Fig. 18-21. For the specified notch frequency we can draw the line in the $(p, q)$ grid with the slope corresponding to the notch frequency and assess the optimal integer values $p$, $q$ as demonstrated in Fig. 18-21. For the FIR MF notch filter we can express the relation between the integer values $q$ and $p$ based on (50) as follows

$$q = \frac{1 + \cos \omega_mT}{1 - \cos \omega_mT} p \ .$$

(55)

The relation (55) defines in the $(p, q)$ plane the line with the slope $(1 + \cos \omega_mT)/(1 - \cos \omega_mT)$. In Fig. 18 and Fig. 19 the relation (55) is represented by the solid lines. The distance of the particular discrete point $(p, q)$ from this line expresses precisely the deviation of the actual and the specified notch frequency. This is demonstrated in the comments on our first and second example.

Comment on example No. 1

In our first example the deviation of the actual and the specified notch frequency amounts 0.0002$\pi$. From Fig. 18 we can conclude that the calculated integer values $p = 12$, $q = 32$ represent an excellent solution with respect to the filter specification. No reasonable alternative in terms of $(p, q)$ values for the filter specification is available.

Comment on example No. 2

In our second example the filter specification was $\omega_mT = 0.84\pi$ and $\Delta \omega T = 0.122\pi$ for $\alpha = -3.0103$ dB. Using our design procedure we obtained $p = 63$, $q = 4$ ($N = 135$) resulting in $\omega_mT = 0.8429\pi$ and $\Delta \omega T = 0.1207\pi$. The obtained notch frequency deviates from the specified value by 0.0029$\pi$. Due to the discrete nature of the notch frequency (50) we can see in Fig. 19 that an actual notch frequency is closer to the specified value. It can be achieved for $p = 76$, $q = 5$ ($N = 163$) which yields $\omega_mT = 0.8402\pi$ and $\Delta \omega T = 0.1099\pi$. The notch frequency obtained deviates from the specified value by 0.0002$\pi$. The FIR MF notch filter fulfills a stricter specification than required. For $p = 61$, $q = 4$
For the FIR ER notch filter we can express the relation between \( q \) and \( p \) based on (51) as follows

\[
q = \left( \frac{\pi}{\omega_m(0)T} - 1 \right) p.
\]

The relation (56) defines the line in the \((p, q)\) plane with the slope \( \pi / (\omega_m(0)T) - 1 \). In Fig. 20-21 the relation (56) is represented by the solid lines. However, unlike the FIR MF notch filter the distance of the discrete point \((p, q)\) from the line defined by (56) may not express exactly the deviation of the actual and the specified notch frequency of the FIR ER notch filter. This is because of the possible slight deviation of the actual notch frequency \( \omega_m(\kappa)T \) and its limit value \( \omega_m(0)T \) used in (56). However, the inaccuracy is negligible because the particular value \( \omega_m(\kappa)T \) deviates only slightly from its limit value \( \omega_m(0)T \), as explained in the previous section. Thus although not exactly precise, the distance of the discrete point \((p, q)\) from the line defined by (56) represents an excellent measure of the deviation of the actual and the specified notch frequency of the FIR ER notch filter. This is demonstrated in the comments on our third and fourth examples.

Comment on example No. 3
In our third example the filter specification was \( \omega_mT = 0.35 \pi \) and \( \Delta\omega T = 0.075 \pi \) for \( a = -1 \) dB. The designed filter based on \( p = 11, q = 20, \kappa = 0.641747 \) \((N = 63)\) resulted in \( \omega_mT = 0.3554 \pi, \Delta\omega T = 0.0756 \pi \) and \( a = -0.8647 \) dB. The actual notch frequency deviates from the specified value by 0.0054 \( \pi \). From Fig. 20 we can see some alternatives yielding the actual notch frequency closer to the specified value. The parameters of the alternative FIR MF notch filters obtained by (49), (48), (39) are summarized in Table VIII.

### Table VIII

<table>
<thead>
<tr>
<th>( p, q )</th>
<th>( N )</th>
<th>( \omega_mT )</th>
<th>( \Delta\omega T )</th>
<th>( a ) [dB]</th>
</tr>
</thead>
<tbody>
<tr>
<td>11, 20</td>
<td>63</td>
<td>0.3554 ( \pi )</td>
<td>0.0756 ( \pi )</td>
<td>-0.8647</td>
</tr>
<tr>
<td>12, 22</td>
<td>69</td>
<td>0.3535 ( \pi )</td>
<td>0.0754 ( \pi )</td>
<td>-0.6120</td>
</tr>
<tr>
<td>13, 24</td>
<td>75</td>
<td>0.3519 ( \pi )</td>
<td>0.0752 ( \pi )</td>
<td>-0.4246</td>
</tr>
<tr>
<td>14, 26</td>
<td>81</td>
<td>0.3505 ( \pi )</td>
<td>0.0751 ( \pi )</td>
<td>-0.3067</td>
</tr>
</tbody>
</table>

Comment on example No. 4
In our fourth example the filter specification was \( \omega_mT = 0.84 \pi \) and \( \Delta\omega T = 0.061 \pi \) for \( a = -0.95 \) dB. The designed filter based on \( p = 32, q = 6, \kappa = 0.743599 \) \((N = 77)\) resulted in \( \omega_mT = 0.8408 \pi, \Delta\omega T = 0.0607 \pi \) and \( a = -0.9109 \) dB. The actual notch frequency deviates from the specified value by 0.0008 \( \pi \). From Fig. 21 we can assess, that an actual notch frequency is closer to the specified value. It can be achieved for \( p = 37, q = 7 \) \((N = 89)\). Indeed, the filter parameters are \( \omega_mT = 0.8396 \pi \) \((49)\) and \( \Delta\omega T = 0.0611 \pi \) \((48)\) for \( a = -0.4950 \) dB \((39)\). The actual notch frequency deviates from the specified value by 0.0004 \( \pi \).

XIII. APPENDIX - ALGEBRAIC EVALUATION OF THE JACOBI’S ZETA FUNCTION AND OF THE ELLIPTIC INTEGRAL OF THE THIRD KIND

In the design of FIR ER notch filter we need to evaluate the Jacobi’s Zeta function and the elliptic integral of the
The Jacobi’s Zeta function is usually evaluated numerically by the arithmetic-geometric mean [1]. For the elliptic integral of the third kind a similar algorithm based on arithmetic-geometric mean is not known. For the evaluation of the Jacobi’s Zeta function and of the elliptic integral of the third kind we propose to use the algebraic algorithms derived in [13], [14].

Jacobi’s Zeta function

In [13], [14] the algebraic algorithm for the evaluation of the Jacobi’s Zeta function \( Z(\frac{\pi}{n} K(k)|\kappa) \) of the “discrete” argument in form of Matlab function \( u = \text{zeta}(n,k) \) was presented.

```matlab
function u = zeta(n,k)
% zeta(n,k)
% * Jacobi’s Zeta Function of discrete
% * argument K(k)/n
% * evaluation based on addition theorem
% * Z(u) + Z(v) - Z(u+v) =
% * k*k*sn(u(k))*sn(v(k))*sn(u+v(k))
% % Erlangen, June 1997, Mironslav Vlcek
% % *******************************************************************************
quarter=ellipke(k.*k);
s=ellipj((1:n)*quarter/n, k.*k);
v=s(n-(1:n-1)).*s(n+1-(1:n-1));
a=diag(n-1:-1:1).*ones(n-1));
b=ones(n-1)-tril(ones(n-1));
u=k.*k*s(1)/n.*(a-n*b).*v';

% *******************************************************************************

The “discrete” argument denotes the integer multiples of the real value \( \frac{\pi}{n} K(k) \), where \( K(k) \) is the real valued quarter-period and \( p, n \) are integer values. The function \( u = \text{zeta}(n,k) \) has two arguments, the integer value \( n \) and the Jacobi’s elliptic modulus \( \kappa \). The function \( u = \text{zeta}(n,k) \) returns the column vector \( u \) containing \( n-1 \) values of the Zeta function of the discrete argument \( Z(\frac{\pi}{n} K(k)|\kappa) \) for \( p = 1 ... n-1 \). For reference we will demonstrate the usage of the function \( u = \text{zeta}(n,k) \) in connection with our third example. When calculating the actual value of the notch frequency \( \omega_{\text{not}} T = 49 \) (using (32)) or the attenuation in the passbands \( a \) [dB] using (37), (39), the integer values \( p, n \) are available and the Matlab code for the evaluation of the particular value \( v = Z(\frac{\pi}{n} K(k)|\kappa) \) may look as follows

```matlab
% values resulting from example No. 3
% integer values p=11, n=31
% k = 0.641747 ( elliptic modulus \kappa \) ; % is the row vector of n-1 terms
u = zeta(n,k)'; % p-th term of the vector u
v = u(p); % represents the requested value
```

However, when evaluating the degree \( n \) using the degree equation (35) the integer values \( p, n \) are not available. From the partition equation (33) and from (34) we obtain the ratio \( r = \frac{n}{p} \) only. In this case we have to evaluate the Zeta function of “continuous” argument \( v = Z(r K(k)|\kappa) \). Even for the evaluation of the Zeta function of the continuous argument we can use the algebraic procedure \( u = \text{zeta}(n,k) \) for the discrete argument. We can expect results with excellent accuracy comparable with those obtained by the numerical arithmetic-geometric mean. We will demonstrate the usage of the function \( u = \text{zeta}(n,k) \) for the continuous argument in connection with our third example. The equations (33), (34) yield the ratio \( r = \frac{n}{p} = 0.3494 \). The elliptic modulus is \( \kappa = 0.641747 \). The Matlab code for the evaluation of the particular value \( v = Z(r K(k)|\kappa) \) may look as follows

```matlab
% values resulting from example No. 3
% r = 0.349434 ( r=p/n )
% k = 0.641747 ( elliptic modulus \kappa \) ; % row vector of 999 terms
p = round(1000*r); % nearest integer nominator
u = zeta(1000,k)'; % p-th term of the vector u
v = u(p); % represents the requested value
```

The result is \( v = 0.106901 \). By the arithmetic-geometric [1] mean with chosen degree of accuracy \( \epsilon = 10^{-12} \) we obtain \( v = 0.106967 \). From our experience, the discretization of the continuous argument to 1000 discrete points gives an excellent
accuracy. The plot of the Zeta function \( Z \left( \frac{\pi}{n}, K(k) \right) \) for our third example is shown in Fig. 22. The source Matlab code is

```matlab
k=0.641747; % \kappa in example No. 3
v=zeta(1000,k'); % algebraic calculation
plot(p,v,'LineWidth',2), grid on, xlabel('p'),
```

Fig. 22. Jacobi’s Zeta function \( Z \left( \frac{\pi}{n}, K(k) \right) \) for \( \kappa = 0.641747 \), \( n = 1000 \), \( p = 1 \ldots n - 1 \) used in example No. 3.

Elliptic integral of the third kind

In [13], [14] the algebraic algorithm for the evaluation of the elliptic integral of the third kind \( \Pi \left( u, \frac{\pi}{n}, K(k) \right) \) of the “discrete” argument in form of Matlab function \( f=ellipi(u,n,k) \) was presented.

```matlab
function f=ellipi(u,n,k)
%******************************************************************************
% * f=ellipi(u,n,k) *
% * Elliptic integral of the third kind *
% * of discrete parameter K(k)/n, *
% * argument u and modulus k *
% * evaluation based on addition theorem *
% * for parameters *
% * P(u,a) + P(u,b) - P(u,a+b) = R(u,a,b) *
% * Erlangen, July 1997, Miroslav Vlcek *
%******************************************************************************
quarter=ellipke(k*k);
si=ellipj(1:n)*quarter/n,k*k);
sp=ellipj((1:n)*quarter/n+u,k*k);
sm=ellipj(1:n)*quarter/n-u,k*k);
v=u+k*k*s1)*s(n-1:n-1)*s(n+1:n-1); nu=1+k*k<s1)*s(n-1:n-1)*s(n+1:n-1));
de=1+k*k<s1)*s(n-1:n-1)*sp(n+1:n-1);
r=log(nu/de)/2+v;

% values resulting from example No. 3
% integer values p=11, n=31
% k = 0.641747 (elliptic modulus \kappa)
% sm = 0.875752 (aux. parameter \sigma_{m})
% r=f(sm,1000,k'); % row vector of n-1 terms
% p=round(1000*r); % p-th term of the vector
% y=f(p); % the p-th term of the vector

r=f(sm,1000,k'); % row vector of n-1 terms
p=round(1000*r); % p-th term of the vector
v=f(p); % the p-th term of the vector

% Demonstration of the third elliptic integral
k=0.641747; % \kappa in example No. 3
v=zeta(1000,k'); % algebraic calculation
plot(p,v,'LineWidth',2), grid on, xlabel('p'),
```

returns the column vector containing \( n - 1 \) values of the elliptic integral of the third kind of the discrete argument. For reference we will demonstrate the usage of the function \( f=ellipi(u,n,k) \) in connection with our third example. When calculating the actual value of the attenuation in the passbands \( a \) [dB] using the maximum value \( y_m \) (37), the integer values \( p, n \) are available and the Matlab code for the evaluation of the particular value \( v = \Pi \left( \sigma_m, \frac{\pi}{n}, K(k) \right) \) may look as follows

```matlab
% values resulting from example No. 3
% integer values p=11, n=31
% k = 0.641747 (elliptic modulus \kappa)
% sm = 0.875752 (aux. parameter \sigma_{m})
% r=f(sm,1000,k'); % row vector of n-1 terms
% p=round(1000*r); % p-th term of the vector
% y=f(p); % the p-th term of the vector

p=round(1000*r); % p-th term of the vector
v=f(sm,p); % the p-th term of the vector

% Demonstration of the third elliptic integral
k=0.641747; % \kappa in example No. 3
v=zeta(1000,k'); % algebraic calculation
plot(p,v,'LineWidth',2), grid on, xlabel('p'),
```

However, when evaluating the degree \( n \) using the degree equation (35) the integer values \( p, n \) are not available. From the partition equation (33) and from (34) we obtain the ratio \( r = \frac{p}{n} \) only. In this case we have to evaluate the elliptic integral of the third kind of “continuous” argument \( f = \Pi(u, f(K(k)) \). Even for its evaluation we can use the algebraic procedure \( f=ellipi(u,n,k) \) for the discrete argument. We will obtain results with excellent accuracy more reliable compared to those obtained by the numerical integration of the defining integral for the elliptic integral of the third kind [1]. We will demonstrate the usage of the function \( f=ellipi(u,n,k) \) for the continuous argument in connection with our third example. The equations (33), (34) yield the ratio \( r = \frac{p}{n} = 0.34941 \). The auxiliary parameter \( \sigma_m = 0.875752 \) follows from (36) and the elliptic modulus \( k \) is \( 0.641747 \). The Matlab code for the evaluation of the particular value \( v = \Pi \left( \sigma_m, \frac{\pi}{n}, K(k) \right) \) may look as follows

```matlab
% Demonstration of the third elliptic integral
k=0.641747; % \kappa in example No. 3
v=zeta(1000,k'); % algebraic calculation
plot(p,v,'LineWidth',2), grid on, xlabel('p'),
```

XIV. CONCLUDING REMARKS

In our paper we have presented a purely analytical solution of the design of the FIR maximally flat and FIR equiripple notch filters. The degree formulae were presented. The recursive algorithms were derived leading directly to the impulse response coefficients of both filter types. We have shown the limit relation between both filter types. We have emphasized the discrete nature of the notch frequency and its impact on the filter design. The algebraic evaluation of the Jacobi's Zeta function is consistent with the parameter values. The equations (33), (34) yield the ratio \( r = \frac{p}{n} = 0.34941 \). The auxiliary parameter \( \sigma_m = 0.875752 \) follows from the partition equation (36) and the elliptic modulus \( k \) is \( 0.641747 \). The Matlab code for the evaluation of the particular value \( v = \Pi \left( \sigma_m, \frac{\pi}{n}, K(k) \right) \) may look as follows

```matlab
% Demonstration of the third elliptic integral
k=0.641747; % \kappa in example No. 3
v=zeta(1000,k'); % algebraic calculation
plot(p,v,'LineWidth',2), grid on, xlabel('p'),
```
Fig. 23. Elliptic integral of the third kind \( \Pi \left( \sigma_m, p, \frac{K(\kappa)}{n} \right) \) for \( \sigma_m = 0.875752, \kappa = 0.641746, n = 1000, p = 1 \ldots n - 1 \) used in example No. 3.

function and of the elliptic integral of third kind used in the filter design was demonstrated.

REFERENCES