# An Analytical Procedure for Critical Frequency Tuning of FIR Filters

Pavel Zahradník and Miroslav Vlček

***Abstract*—A novel analytical procedure for the tuning of finite-impulse response (FIR) filters is introduced. The tuning procedure adjusts a single frequency of the frequency response to the desired value while preserving the nature of the filter. The impulse responses of the original and of the final filter are related by the transformation matrix. Two examples in the analytical design of notch FIR filters demonstrate the usefulness of the proposed tuning procedure.**

***Index Terms*—Equiripple filter, finite-impulse response (FIR) filter, maximally flat filter, notch filter, tuning.**

## I. INTRODUCTION

PRECISE tuning of the frequency properties is an useful operation in the design of digital filters. It can replace the design of the filter from the scratch by the reusing of the impulse response of the available filter. Adaptive filtering is one of the applications. Further, the tuning is useful in the analytical design of digital FIR filters where the available critical frequencies are usually quantized [1], [2], [3]. This quantization prevents such analytical procedures from the design of filters with arbitrarily specified critical frequencies. Hence the analytical design combined with the tuning of the filter represents a powerful design tool. In this brief we present a fast versatile tuning procedure which adjusts a single frequency of the frequency response of the FIR filter to the specified value while preserving the nature of the filter, e.g. maximally flat, equiripple etc. Our tuning procedure is based on the expansion of the Chebyshev polynomial of the transformed argument into the sum of Chebyshev polynomials resulting in the transformation matrix. The impulse response of the final filter is obtained from the impulse response of the original filter by applying of the transformation matrix.

## II. ZERO PHASE TRANSFER FUNCTION

We assume the impulse response $h(k)$ with odd length $N = 2n+1$ and with even symmetry

$$a(0) = h(n)\ ,\ \ a(k) = 2h(n+k) = 2h(n-k)\,,\ k = 1 \ldots n\,. \tag{1}$$

The vector $a(k)$ is more useful for further manipulations than the corresponding impulse response $h(k)$. For brevity we call $a(k)$ the $a$-vector of the filter. The transfer function of the filter is

$$\begin{aligned} H(z) &= \sum_{k=0}^{2n} h(k)\, z^{-k} \\ &= z^{-n}\left[h(n)\ + 2\sum_{k=1}^{n} h(n\pm k)\,\frac{1}{2}\left(z^{k}+z^{-k}\right)\right] \\ &= z^{-n}\sum_{k=0}^{n} a(k)\,T_k(w) = z^{-n}Q(w) \end{aligned} \tag{2}$$

where $T_k(w)$ is the Chebyshev polynomial of the first kind. The function

$$Q(w) = \sum_{k=0}^{n} a(k)\,T_k(w) \tag{3}$$

represents a polynomial in the variable $w = \frac{1}{2}(z+z^{-1})$ which on the unit circle $z = e^{j\omega T}$ reduces to the real valued zero phase transfer function (ZPTF) $Q(w)$ of the real argument

$$w = \cos(\omega T)\,. \tag{4}$$

## III. DIFFERENTIAL EQUATIONS

The Chebyshev polynomial of the first kind $T_k(x)$ fulfills the differential equation

$$(1-x^2)\frac{d^2T_k(x)}{dx^2} - x\frac{dT_k(x)}{dx} + k^2T_k(x) = 0\ . \tag{5}$$

We have derived the differential equation

$$\begin{aligned} &(1-w^2+2\frac{\lambda^{'}}{\lambda}(1-w))\frac{d^2F_+(w)}{dw^2} \\ &\qquad -(w+\frac{\lambda^{'}}{\lambda})\frac{dF_+(w)}{dw} + k^2F_+(w) = 0 \end{aligned} \tag{6}$$

for the polynomial

$$F_+(w) = T_k(\lambda w + \lambda^{'}) \tag{7}$$

and the differential equation

$$\begin{aligned} &(1-w^2+2\frac{\lambda^{'}}{\lambda}(1+w))\frac{d^2F_-(w)}{dw^2} \\ &\qquad -(w-\frac{\lambda^{'}}{\lambda})\frac{dF_-(w)}{dw} + k^2F_-(w) = 0 \end{aligned} \tag{8}$$

for the polynomial

$$F_-(w) = T_k(\lambda w - \lambda^{'}) \tag{9}$$

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M. Vlček is with the Czech Technical University in Prague, Faculty of Transportation Sciences, Department of Applied Mathematics, Konviktská 20, CZ-110 00 Praha 1, email : vlcek@fd.cvut.cz

P. Zahradník is with the Czech Technical University in Prague, Faculty of Electrical Engineering, Department of Telecommunications Engineering, Technická 2, CZ-166 27 Praha 6, email : zahradni@fel.cvut.cz

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TABLE I
RECURSIVE ALGORITHM FOR THE EVALUATION OF THE COEFFICIENTS $\alpha_k(m)$ OF THE TRANSFORMATION MATRIX $A_+$.

| | |
|---|---|
| *given* | $k$ (integer value), $0 < \lambda \le 1$ (real value) |
| *initialization* | $\lambda' = 1 - \lambda$<br>$\alpha_k(k+1) = \alpha_k(k+2) = \alpha_k(k+3) = 0$<br>$\alpha_k(k) = \lambda^k$ |
| *body*<br>(*for* $\mu = -3 \ldots k-4$ ) | $\alpha_k(k-\mu-4) =$<br>$\{$<br>$-2\left[(\mu+3)(2k-\mu-3) - \frac{\lambda'}{\lambda}(k-\mu-3)(2k-2\mu-7)\right]\alpha_k(k-\mu-3)$<br>$+2\frac{\lambda'}{\lambda}(k-\mu-2)\ \alpha_k(k-\mu-2)$<br>$+2\left[(\mu+1)(2k-\mu-1) - \frac{\lambda'}{\lambda}(k-\mu-1)(2k-2\mu-1)\right]\alpha_k(k-\mu-1)$<br>$+\mu(2k-\mu)\ \alpha_k(k-\mu)$<br>$\}\ /\ (\mu+4)(2k-\mu-4)$ |
| (*end loop on* $\mu$) | |

where the real values $\lambda$ and $\lambda'$ are related by

$$\lambda + \lambda' = 1 \ . \tag{10}$$

## IV. TRANSFORMED ZERO-PHASE TRANSFER FUNCTION

The purpose of the frequency transformation is to map the critical frequency $\omega_m T$ of the frequency response of the filter to the desired value $\omega_0 T$. The mapping $\omega_m T \leftrightarrow \omega_0 T$ in the frequency domain is equivalent to the mapping $w_m \leftrightarrow w_0$ in the $w$-domain. Due to (4) the shift in both domains occurs in opposite directions. We propose the transformed ZPTFs in the form

$$x = \lambda w + \lambda' \ , \ \ \lambda = \frac{w_m - 1}{w_0 - 1} \tag{11}$$

if $\omega_m T < \omega_0 T$ and

$$x = \lambda w - \lambda' \ , \ \ \lambda = \frac{w_m + 1}{w_0 + 1} \tag{12}$$

if $\omega_0 T < \omega_m T$. The procedure provides the impulse response coefficients for a FIR filter with following properties.

- The frequency $\omega_m T$ to the specified value $\omega_0 T$ is adjusted.
- The maximal attenuation in the passband(s) and the minimal attenuation of the stopband(s) of the filter is preserved.
- The width of the bands of the filter is broadened.

The tranformed ZPTFs

$$Q_t(w) = \sum_{k=0}^{n} a(k)\, T_k(x) = \sum_{k=0}^{n} a(k) \sum_{m=0}^{k} \alpha_k(m) T_m(w) \tag{13}$$

can be rewritten in the matrix form

$$Q(w) = [a(0)\ \ a(1)\ \ \cdots\ \ a(n)] \times \tag{14}$$

$$\begin{bmatrix} \alpha_0(0) & 0 & 0 & 0 & \cdots & 0 \\ \alpha_1(0) & \alpha_1(1) & 0 & 0 & \cdots & 0 \\ \alpha_2(0) & \alpha_2(1) & \alpha_2(2) & 0 & \cdots & 0 \\ \alpha_3(0) & \alpha_3(1) & \alpha_3(2) & \alpha_3(3) & \cdots & 0 \\ \vdots & & & & & \vdots \\ \alpha_n(0) & \alpha_n(1) & \alpha_n(2) & \alpha_n(3) & \cdots & \alpha_n(n) \end{bmatrix} \times \begin{bmatrix} T_0(w) \\ T_1(w) \\ T_2(w) \\ T_3(w) \\ \vdots \\ T_n(w) \end{bmatrix}$$

$$= a\ \ A\ \ \mathcal{T} \ . \tag{15}$$

We call the low triangular matrix $A$ the transformation matrix. The vector $a_t$ of the transformed filter is given by the product of the vector $a$ of the original filter and the transformation matrix $A$

$$a_t = a\ \ A \ . \tag{16}$$

There are two transformation matrices $A_+$ and $A_-$ corresponding to the transformations (11) and (12). The fast evaluation of the coefficients $\alpha_k(m)$ of the transformation matrices is essential in the adaptive filtering. Our evaluation procedure results from the differential equations (6), (8) of the corresponding polynomials (7), (9).

## V. EVALUATION OF THE TRANSFORMATION MATRIX

Based on the differential equations (6) and (8) we have derived fast procedure for the evaluation of the coefficients $\alpha_k(m)$ of the transformation matrices $A_+$ and $A_-$. The fast algorithm for the evaluation of the coefficients of the transformation matrix $A_+$ is summarized in Tab. I. Its derivation is presented in Appendix. The evaluation of the transformation matrix $A_-$ is analogical. Both matrices differs by the signs of the "odd" coefficients $\alpha_k(k-\mu-3)$ and $\alpha_k(k-\mu-1)$ only - see Tab. II.

TABLE II
RECURSIVE ALGORITHM FOR THE EVALUATION OF THE COEFFICIENTS $\alpha_k(m)$ OF THE TRANSFORMATION MATRIX $A_-$.

*given* $k$ (integer value), $0 < \lambda \le 1$ (real value)

*initialization*

$$\lambda' = 1 - \lambda$$
$$\alpha_k(k+1) = \alpha_k(k+2) = \alpha_k(k+3) = 0$$
$$\alpha_k(k) = \lambda^k$$

*body*
(*for* $\mu = -3 \ldots k-4$ )

$$\alpha_k(k-\mu-4) = \Big\{ +2\left[(\mu+3)(2k-\mu-3) - \frac{\lambda'}{\lambda}(k-\mu-3)(2k-2\mu-7)\right]\alpha_k(k-\mu-3) + 2\frac{\lambda'}{\lambda}(k-\mu-2)\;\alpha_k(k-\mu-2) - 2\left[(\mu+1)(2k-\mu-1) - \frac{\lambda'}{\lambda}(k-\mu-1)(2k-2\mu-1)\right]\alpha_k(k-\mu-1) + \mu(2k-\mu)\;\alpha_k(k-\mu) \Big\} \;/\; (\mu+4)(2k-\mu-4)$$

(*end loop on* $\mu$)

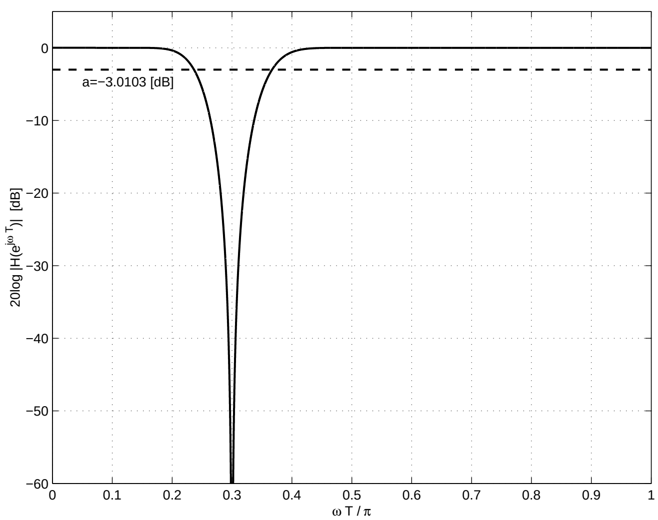


Fig. 1. Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB].

## VI. EXAMPLES OF TUNING

The proposed tuning procedure represents a versatile design tool. It is especially useful in the design of notch FIR filters as a notch filter is primarily specified by one critical frequency which is the notch frequency. In [3] we have shown that the available notch frequencies are quantized in the analytical design of maximally flat and equiripple notch FIR filters. This drawback can be eliminated by the proposed tuning procedure as shown in our examples.

*Example 1:* Design the maximally flat notch FIR filter specified by the notch frequency $\omega_0 T = 0.3\,\pi$ and width of the notchband $\Delta\omega T = 0.13\,\pi$ for the maximal attenuation in the passbands $a = -3.0103$ dB.

Using the analytical design procedure [3] we get $n = 59$, $p = 12$ and $q = 47$. The designed filter of the length $N = 119$ coefficients with "quantized" notch frequency $\omega_m T = 0.2979\,\pi$ and $\Delta\omega T = 0.1293\,\pi$ for $a = -3.0103$ dB will be tuned using the proposed tuning procedure in order to get the specified notch frequency $\omega_0 T = 0.3\,\pi$. Because of $\omega_m T < \omega_0 T$ we evaluate (Tab. 1) the transformation matrix $A_+$ for $\lambda = 0.9868$ (11). We get the tuned filter with parameters $\omega_0 T = 0.3\,\pi$ and $\Delta\omega T = 0.1304\,\pi$ for $a = -3.0103$ dB. The actual width of the notchband exceeds by $0.28\%$ the specified value. The impulse response $h(k)$ of the "quantized" filter and the impulse response $h_t(k)$ of

TABLE III
COEFFICIENTS OF THE IMPULSE RESPONSES

| k | | h(k) | $h_t(k)$ |
|---|---|---|---|
| 24 | 94 | -0.000002 | -0.000002 |
| 25 | 93 | -0.000002 | -0.000002 |
| 26 | 92 | 0.000002 | 0.000000 |
| 27 | 91 | 0.000012 | 0.000009 |
| 28 | 90 | 0.000023 | 0.000021 |
| 29 | 89 | 0.000020 | 0.000023 |
| 30 | 88 | -0.000024 | -0.000012 |
| 31 | 87 | -0.000112 | -0.000093 |
| 32 | 86 | -0.000177 | -0.000171 |
| 33 | 85 | -0.000081 | -0.000115 |
| 34 | 84 | 0.000281 | 0.000201 |
| 35 | 83 | 0.000758 | 0.000687 |
| 36 | 82 | 0.000819 | 0.000863 |
| 37 | 81 | -0.000152 | 0.000072 |
| 38 | 80 | -0.002046 | -0.001752 |
| 39 | 79 | -0.003358 | -0.003297 |
| 40 | 78 | -0.001783 | -0.002222 |
| 41 | 77 | 0.003425 | 0.002634 |
| 42 | 76 | 0.009021 | 0.008551 |
| 43 | 75 | 0.008551 | 0.009125 |
| 44 | 74 | -0.001977 | -0.000433 |
| 45 | 73 | -0.017327 | -0.016011 |
| 46 | 72 | -0.023187 | -0.023566 |
| 47 | 71 | -0.007622 | -0.009884 |
| 48 | 70 | 0.023666 | 0.021252 |
| 49 | 69 | 0.044852 | 0.044606 |
| 50 | 68 | 0.029660 | 0.032152 |
| 51 | 67 | -0.019971 | -0.016816 |
| 52 | 66 | -0.065671 | -0.064721 |
| 53 | 65 | -0.061489 | -0.063519 |
| 54 | 64 | 0.000557 | -0.002332 |
| 55 | 63 | 0.073377 | 0.072340 |
| 56 | 62 | 0.091085 | 0.092339 |
| 57 | 61 | 0.030758 | 0.032318 |
| 58 | 60 | -0.060246 | -0.060170 |
| 59 | | 0.896722 | 0.895847 |

TABLE IV
COEFFICIENTS OF THE IMPULSE RESPONSES.

| k | | h(k) | $h_t(k)$ |
|---|---|---|---|
| 0 | 72 | 0.016832 | 0.011622 |
| 1 | 71 | 0.004953 | -0.005198 |
| 2 | 70 | -0.002260 | -0.009660 |
| 3 | 69 | -0.009076 | -0.010249 |
| 4 | 68 | -0.008700 | -0.003681 |
| 5 | 67 | 0.000117 | 0.006768 |
| 6 | 66 | 0.010632 | 0.013012 |
| 7 | 65 | 0.013100 | 0.008921 |
| 8 | 64 | 0.003678 | -0.003396 |
| 9 | 63 | -0.010795 | -0.013938 |
| 10 | 62 | -0.017533 | -0.012654 |
| 11 | 61 | -0.009034 | 0.001287 |
| 12 | 60 | 0.009012 | 0.017271 |
| 13 | 59 | 0.021195 | 0.021210 |
| 14 | 58 | 0.015517 | 0.007859 |
| 15 | 57 | -0.004980 | -0.013249 |
| 16 | 56 | -0.023239 | -0.024485 |
| 17 | 55 | -0.022375 | -0.014929 |
| 18 | 54 | -0.001244 | 0.009158 |
| 19 | 53 | 0.022942 | 0.028084 |
| 20 | 52 | 0.028636 | 0.024800 |
| 21 | 51 | 0.009196 | 0.000144 |
| 22 | 50 | -0.019871 | -0.026360 |
| 23 | 49 | -0.033269 | -0.032071 |
| 24 | 48 | -0.018035 | -0.010712 |
| 25 | 47 | 0.014008 | 0.021071 |
| 26 | 46 | 0.035383 | 0.036681 |
| 27 | 45 | 0.026667 | 0.021933 |
| 28 | 44 | -0.005787 | -0.012097 |
| 29 | 43 | -0.034396 | -0.037428 |
| 30 | 42 | -0.033926 | -0.032360 |
| 31 | 41 | -0.003929 | -0.000239 |
| 32 | 40 | 0.030153 | 0.032628 |
| 33 | 39 | 0.038727 | 0.038693 |
| 34 | 38 | 0.013946 | 0.012424 |
| 35 | 37 | -0.023124 | -0.024678 |
| 36 | | 0.933816 | 0.932507 |

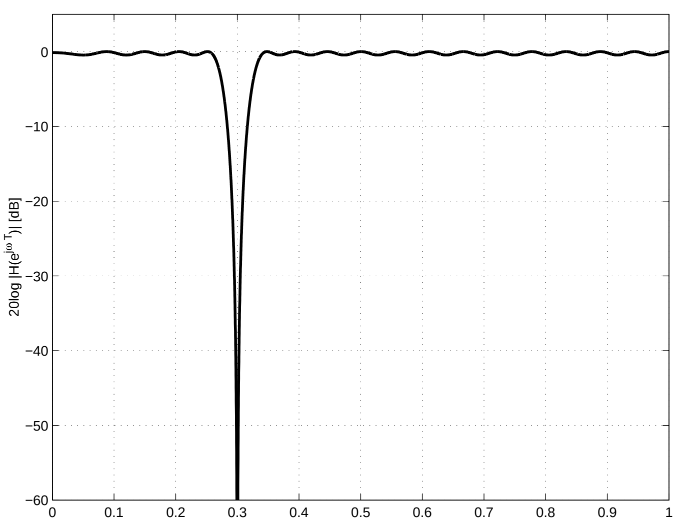


Fig. 2. Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB].

the tuned filter are summarized in Table III. Because of the marginal coefficients of both impulse responses are less than $10^{-6}$ for $k < 24$ and $k > 94$, only the 71 central coefficients are presented. The amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB] of the tuned maximally flat notch FIR filter is shown in Fig. 1.

*Example 2:* Design the equiripple notch FIR filter specified by the notch frequency $\omega_0 T = 0.3\,\pi$, width of the notchband $\Delta\omega T = 0.075\,\pi$ for the maximal attenuation in the passbands $a = -0.5$ dB.

Using the analytical design procedure [3] we get $\kappa = 0.665619$, $n = 36$ , $p = 11$ and $q = 25$. The designed filter of the length $N = 73$ coefficients with "quantized" notch frequency $\omega_m T = 0.3064\,\pi$ and $\Delta\omega T = 0.075\,\pi$ for $a_{act} = -0.4584$ dB will be tuned using the proposed tuning procedure in order to get the specified notch frequency $\omega_0 T = 0.3\,\pi$. Because $\omega_0 T < \omega_m T$ we evaluate (Sec. V) the transformation matrix $A_-$ for $\lambda = 0.9898$ (12). We get the tuned filter with parameters $\omega_0 T = 0.3\,\pi$ and $\Delta\omega T = 0.0779\,\pi$ for $a = -0.4584$ dB. The actual width of the notchband exceeds by $3.99\%$ the specified value. The impulse response $h(k)$ of the "quantized" filter and the impulse response $h_t(k)$ of the tuned filter are summarized in Table IV. The amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB] of the tuned equiripple notch FIR filter is shown in Fig. 2. The detailed view of the passbands of the "quantized" and of the tuned filter is shown in Fig. 3.

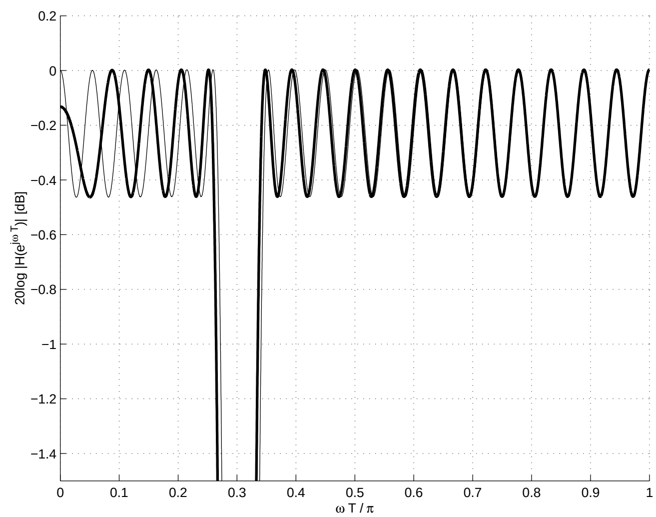


Fig. 3. Passbands of the "quantized" (thin line) and of the tuned filter.

## VII. CONCLUDING REMARKS

In this paper we have presented a fast analytical tuning procedure for FIR filters. The analytical tuning procedure is based on the differential equation of the transformed Chebyshev polynomial. Two examples demonstrate the efficiency of the tuning procedure in the design of FIR filters.

## VIII. APPENDIX - DERIVATION OF THE ALGORITHM

In order to derive the recursive algorithm (Tab. I) for the evaluation of the coefficients $\alpha_k(m)$ of the transformation matrix $A_+$ we insert the polynomial

$$F_+(w) = T_k(\lambda w + \lambda') = \sum_{m=0}^{k} \alpha_k(m) T_m(w) \qquad (17)$$

into the differential equation (6). It yields

$$\sum_{m=0}^{k}\alpha_k(m)(k^2-m^2)T_m(w) \qquad (18)$$
$$+\frac{\lambda'}{\lambda}\sum_{m=0}^{k}\alpha_k(m)\left[2(1-w)\frac{d^2T_m(w)}{dw^2}-\frac{dT_m(w)}{dw}\right]=0\ .$$

Because of

$$2T_m(w)=U_m(w)-U_{m-2}(w) \qquad (19)$$
$$2wU_{m-1}(w)=U_m(w)+U_{m-2}(w) \qquad (20)$$

we obtain

$$\sum_{m=0}^{k}\alpha_k(m)\frac{k^2-m^2}{2}\times \qquad (21)$$
$$[U_{m+1}(w)+2U_m(w)-2U_{m-2}(w)-U_{m-3}(w)]$$
$$+\frac{\lambda'}{\lambda}\sum_{m=0}^{k}\alpha_k(m)\,m\times$$
$$[-(2m-1)U_m(w)-2U_{m-1}(w)+(2m+1)U_{m-2}(w)]=0$$

where $U_m(w)$ is the Chebyshev polynomial of the second kind. We write the equation (21) explicitly as

$$0=\alpha_k(k)\frac{k^2-k^2}{2}\times \qquad (22)$$
$$[U_{k+1}(w)+2U_k(w)-2U_{k-2}(w)-U_{k-3}(w)]$$
$$+\frac{\lambda'}{\lambda}\alpha_k(k)\,k\times$$
$$[-(2k-1)U_k(w)-2U_{k-1}(w)+(2k+1)U_{k-2}(w)]$$
$$+\alpha_k(k-1)\frac{k^2-(k-1)^2}{2}\times$$
$$[U_k(w)+2U_{k-1}(w)-2U_{k-3}(w)-U_{k-4}(w)]$$
$$+\frac{\lambda'}{\lambda}\alpha_k(k-1)\,(k-1)\times$$
$$[-(2k-3)U_{k-1}(w)-2U_{k-2}(w)+(2k-1)U_{k-3}(w)]$$
$$+\alpha_k(k-2)\frac{k^2-(k-2)^2}{2}\times$$
$$[U_{k-1}(w)+2U_{k-2}(w)-2U_{k-4}(w)-U_{k-5}(w)]$$
$$+\frac{\lambda'}{\lambda}\alpha_k(k-2)\,(k-2)\times$$
$$[-(2k-5)U_{k-2}(w)-2U_{k-3}(w)+(2k-3)U_{k-4}(w)]$$
$$+\dots$$

Than, we collect the coefficients associated with the descending degree of the Chebyshev polynomials $U_m(w)$ and requiring that they all are equal zero we obtain the set of identities

$$0=U_{k+1}(w)\times\left[\frac{k^2-k^2}{2}\,\alpha_k(k)\right]$$

$$0=U_k(w)\times$$
$$\left[-\frac{\lambda'}{\lambda}\,k(2k-1)\,\alpha_k(k)+\frac{k^2-(k-1)^2}{2}\,\alpha_k(k-1)\right]$$
$$0=U_{k-1}(w)\times$$
$$\left[-2\frac{\lambda'}{\lambda}\,k\,\alpha_k(k)+(k^2-(k-1)^2)\,\alpha_k(k-1)\right.$$
$$\left.-\frac{\lambda'}{\lambda}(k-1)(2k-3)\,\alpha_k(k-1)+\frac{k^2-(k-2)^2}{2}\,\alpha_k(k-2)\right]$$
$$\vdots$$

The first identity suggests that $\alpha_k(k)$ can be chosen arbitrarily. Due to the the expansion

$$T_k(\lambda w+\lambda')=\lambda^k w^k+\ \dots\ =\lambda^k T_k(w)\ \ +\dots\ \ . \qquad (23)$$

we put

$$\alpha_k(k)=\lambda^k\ . \qquad (24)$$

The other coefficients are evaluated from the above identities recursively as

$$\alpha_k(k-1)=2\frac{\lambda'}{\lambda}\,k\,\alpha_k(k)\ \ . \qquad (25)$$

or

$$\alpha_k(k-2)= \qquad (26)$$
$$-\left[\frac{2k-1}{2k-2}-\frac{\lambda'}{\lambda}\frac{2k-3}{2}\right]\alpha_k(k-1)+\frac{\lambda'}{\lambda}\frac{2k}{2k-2}\alpha_k(k)\ \ .$$

Finally for a general value $m=k-\mu$ we obtain

$$\alpha_k(k-\mu-4)(\mu+4)(2k-\mu-4)= \qquad (27)$$
$$-2\left[(\mu+3)(2k-\mu-3)-\frac{\lambda'}{\lambda}(k-\mu-3)(2k-2\mu-7)\right]\times$$
$$\times\,\alpha_k(k-\mu-3)$$
$$+2\frac{\lambda'}{\lambda}(k-\mu-2)\alpha_k(k-\mu-2)$$
$$+2\left[(\mu+1)(2k-\mu-1)-\frac{\lambda'}{\lambda}(k-\mu-1)(2k-2\mu-1)\right]\times$$
$$\times\,\alpha_k(k-\mu-1)$$
$$+\mu(2k-\mu)\alpha_k(k-\mu)\ \ .$$

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