Zolotarev Polynomials and Optimal FIR Filters

Miroslav Vlček and Rolf Unbehauen, *Fellow, IEEE*

***Abstract*— The algebraic form of Zolotarev polynomials refraining from their parametric representation is introduced. A recursive algorithm providing the coefficients for a Zolotarev polynomial of an arbitrary order is obtained from a linear differential equation developed for this purpose. The corresponding narrow-band, notch and complementary pair FIR filters are optimal in Chebyshev sense. A recursion giving an explicit access to the impulse response coefficients is also presented. Some design examples are included to demonstrate the efficiency of the presented approach.**

***Index Terms*— Zolotarev polynomials, optimal FIR narrow bandpass and notch filters, impulse response, recurrence formula.**

I. INTRODUCTION

DURING the period 1868-1878 E. Zolotarev stated and solved four problems in approximation theory, two concerned of polynomials and the other two of rational functions. The solution of the third problem was introduced in filter theory by W. Cauer in 1933. The first problem concerns of the polynomial of the form

$$f(x) = x^n - n\sigma x^{n-1} + \beta(n-2)x^{n-2} + \ldots + \beta(1)x + \beta(0)\,, \quad (1)$$

which deviates least from zero in a given interval where σ is a given real number. If $\sigma \le \tan^2(\pi/2n)$, the solution is given in terms of Chebyshev polynomials

$$f(x) = \frac{1}{2^{n-1}}(1+\sigma)^n T_n\left(\frac{x-\sigma}{1+\sigma}\right)\,, \quad (2)$$

while for $\sigma > \tan^2(\pi/2n)$ no such solution exists. Zolotarev derived the general solution in terms of elliptic functions. The application of that class of polynomials in filter theory had to wait till 1970 when R. Levy studied odd Achieser-Zolotarev polynomials with the application to the quasi-lowpass filters. In his complete treatise [5] he pointed out that "the most satisfactory method for forming a Zolotarev function would be from closed-form expressions for the coefficients, or from recursion formula. No such formulas have yet been found."

Later, in 1986 X. Chen and T. W. Parks generalised Zolotarev polynomials for the design of optimal FIR narrowband filters exhibiting equiripple behaviour over the stop bands. They have extended the original closed form solution to a more general polynomial and begun to call this extension a Zolotarev polynomial. They have introduced polynomials of degree N with L zeros in the interval (α, β) and $N-L$ zeros in the interval $(-1,1)$ in a standard parametric form

$$x = \frac{sn^2(u|\kappa) + sn^2(\frac{L}{N}\,\mathbf{K}(\kappa)|\kappa)}{sn^2(u|\kappa) - sn^2(\frac{L}{N}\,\mathbf{K}(\kappa)|\kappa)} \quad (3)$$

$$f_{N,L}(u|k) = \frac{(-1)^L}{2}\left[\left(\frac{H(u-\frac{L}{N}\,\mathbf{K}(\kappa))}{H(u+\frac{L}{N}\,\mathbf{K}(\kappa))}\right)^N + \left(\frac{H(u+\frac{L}{N}\,\mathbf{K}(\kappa))}{H(u-\frac{L}{N}\,\mathbf{K}(\kappa))}\right)^N\right], \quad (4)$$

where $H(u - \frac{L}{N}\,\mathbf{K}(\kappa))$ is Jacobi's eta function, $sn(u|\kappa)$ is Jacobi's elliptic function and $\mathbf{K}(\kappa)$ is the complete elliptic integral of the first kind of modulus κ.

Having found satisfactory results using numerical algorithms for the involved special functions - theta, Jacobi's eta and zeta functions X. Chen and T.W. Parks [3] emphasised "E.V. Voronovskaya [11] demonstrated a way to synthesise such polynomials using a linear functional method. In addition, she gave an example of synthesising the Zolotarev polynomial of degree three and derived the analytic formulas for its coefficients. Unfortunately, the general practical algorithm is still not available."

An efficient evaluation of Zolotarev polynomials remains a vivid question in spite of their 120 years history. Recently, in [7] I. W. Selesnick and C. S. Burrus quoted that a subset of maximal ripple bandpass filters can be found using analytic methods involving Zolotarev polynomials as described by X. Chen and T.W. Parks.

In our paper we develop a completely analytic procedure for evaluation of the Zolotarev polynomials [3] which replaces their standard parametric representation. We use a slightly different notation for Zolotarev polynomial $Z_{p,q}(u|\kappa)$ emphasising that p counts the number of zeros right from the maximum and q corresponds to the number of zeros left from the maximum, and $n = p+q$ is the degree. We also introduce the independent variable w which is confined to the intervals $(-1, w_s) \cup (w_p, 1)$ and it is related to the digital domain by

$$w = \frac{1}{2}\,\left(z + z^{-1}\right)\Big|_{z=e^{j\omega T}} = \cos\omega T\,. \quad (5)$$

The intervals $(-1,1) \cup (\alpha, \beta)$ are transformed to the intervals $(-1, w_s) \cup (w_p, 1)$ by the linear transformation of x

$$w = xcn^2(u_0|\kappa) - sn^2(u_0|\kappa)\,, \quad (6)$$

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M. Vlček is with the Czech Technical University, Faculty of Transportation Sciences, Prague, Czech Republic, on leave at the Lehrstuhl für Allgemeine und Theoretische Elektrotechnik, University Erlangen-Nürnberg, Erlangen, Germany.

R. Unbehauen is with the Lehrstuhl für Allgemeine und Theoretische Elektrotechnik, University Erlangen-Nürnberg, Cauerstrasse 7, D-91058 Erlangen, Germany.

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where $u_0 = \dfrac{p}{p+q}\mathbf{K}(\kappa)$. We have derived a linear differential equation from which a recurrent formula for coefficients follow. The algorithm is also extended to the Chebyshev polynomial expansion of Zolotarev's polynomials which is important for direct computation of the impulse response coefficients. Consequently, it replaces the FFT algorithm required in the analytic design of optimal narrow band FIR filters [3].

II. Differential Equation of Approximation and Fundamental Properties

The extremal values of Zolotarev polynomial $Z_{p,q}(u|\kappa)$ of degree $n = p+q$ alternates between -1 and +1 $(p+1)$-times in the interval $(w_p, 1)$ and $(q+1)$-times in the interval $(-1, w_s)$. By inspection the Zolotarev polynomial of Fig.1 satisfies the differential equation

$$(1-w^2)(w-w_p)(w-w_s)\left(\frac{df}{dw}\right)^2 = n^2(1-f^2)(w-w_m)^2\,. \tag{7}$$

This equation (7) expresses the fact that the derivative $\dfrac{df}{dw}$

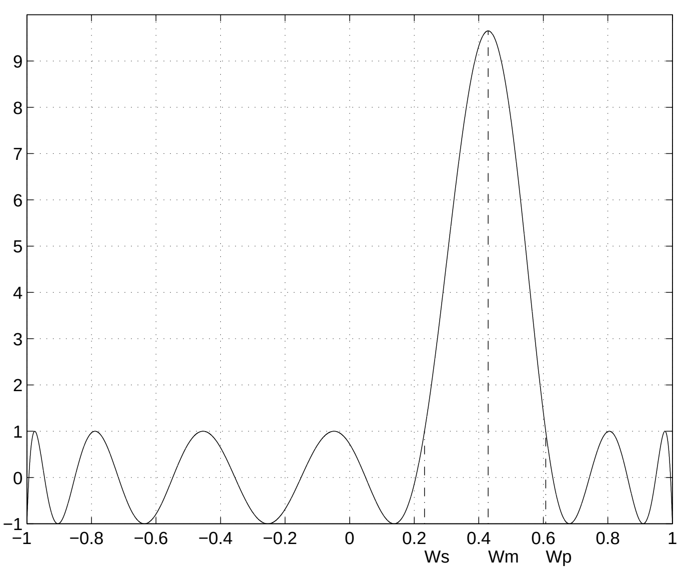

Fig. 1. Zolotarev polynomial $Z_{5,9}(u|0.78)$ of degree 14, with $w_s = 0.2319$, $w_m = 0.4292$ and $w_p = 0.6075$

does not vanish at the points $w = \pm 1, w_s, w_p$ where $f = \pm 1$ for which the right hand side of eq. (7) vanishes, and that $w = w_m$ is a turning point corresponding to the local extrema at which $f \neq \pm 1$. We will call eq. (7) the approximation equation as its form indicates the behaviour of a Zolotarev polynomial. In oder to solve the differential equation (7) we recall conformal transformation [2], [5] from the w plane to the u plane

$$w = \frac{sn^2(u)cn^2(u_0) + cn^2(u)sn^2(u_0)}{sn^2(u) - sn^2(u_0)}\,. \tag{8}$$

Under this transformation the edges w_p and w_s correspond to

$$w_p = 2\,cd^2(u_0|\kappa) - 1 = 2\,sn^2\left(\frac{q}{p+q}\mathbf{K}(\kappa)|\kappa\right) - 1\,, \tag{9}$$

$$w_s = 2\,cn^2(u_0|\kappa) - 1 = 1 - 2\,sn^2\left(\frac{p}{p+q}\mathbf{K}(\kappa)|\kappa\right)\,, \tag{10}$$

while the value w_m is subject to the solution. The conformal transformation (8) suggests the parametrisation in the differential equation (7)

$$\frac{1}{n\sqrt{f^2-1}}\frac{df}{du} = \frac{w - w_m}{\sqrt{(w^2-1)(w-w_p)(w-w_s)}}\frac{dw}{du}\,. \tag{11}$$

Using the inverse transformation to (8)

$$sn^2(u) = sn^2(u_0)\frac{1+w}{w-w_s} \tag{12}$$

and combining with (10) and (11) we obtain

$$\begin{aligned}\frac{dw}{du} &= -4sn(u)cn(u)dn(u)\frac{sn^2(u_0)cn^2(u_0)}{(sn^2(u)-sn^2(u_0))^2} \qquad (13)\\ &= -\frac{dn(u_0)}{sn(u_0)cn(u_0)}\sqrt{(w^2-1)(w-w_p)(w-w_s)}\,.\end{aligned}$$

Then substituting

$$f(w) = \cosh n\Phi \tag{14}$$

equation (11) becomes

$$\begin{aligned}\frac{d\Phi}{du} &= \frac{dn(u_0)}{sn(u_0)cn(u_0)}(w_m - w) \qquad (15)\\ &= \frac{dn(u_0)}{sn(u_0)cn(u_0)}(w_m - w_s)\\ &\qquad -2\,\frac{sn(u_0)cn(u_0)dn(u_0)}{sn^2(u) - sn^2(u_0)}\,.\end{aligned}$$

The eq.(15) can be integrated by using Jacobi's expression [12] for the elliptic integral of the third kind $\Pi(u, u_0|\kappa)$, the theta function $\Theta(u)$ and zeta function $Z(u_0|\kappa)$

$$\Pi(u, u_0|\kappa) = \frac{1}{2}\ln\frac{\Theta(u-u_0)}{\Theta(u+u_0)} + uZ(u_0|\kappa)\,. \tag{16}$$

In view that

$$\frac{\Theta(u - u_0 + i\,\mathbf{K}(\kappa'))}{\Theta(u + u_0 + i\,\mathbf{K}(\kappa'))} = \frac{H(u-u_0)}{H(u+u_0)} \tag{17}$$

we obtain

$$\Phi = u\frac{dn(u_0)}{sn(u_0)cn(u_0)}(w_m - w_s) - \ln\frac{H(u-u_0)}{H(u+u_0)} - 2uZ(u_0|\kappa)\,. \tag{18}$$

Provided we assign the first term to Jacobi's zeta function $Z(u_0)$

$$2Z(u_0) = \frac{dn(u_0)}{sn(u_0)cn(u_0)}(w_m - w_s)\,, \tag{19}$$

it finally reduces to

$$\Phi = \ln\frac{H(u+u_0)}{H(u-u_0)}\,. \tag{20}$$

From eq.(19) the position of the maximum value w_m is found as

$$w_m = w_s + 2\,\frac{sn(u_0)cn(u_0)}{dn(u_0)}Z(u_0)\,. \tag{21}$$

In order to find the argument u_m to which the maximum w_m belongs we write (8) as

$$sn^2(u_m|\kappa) = \frac{w_m + 1}{w_m - w_s} sn^2(\frac{p}{n}\ \mathbf{K}(\kappa)) \tag{22}$$

$$= \frac{sn(\frac{p}{n}\ \mathbf{K})cn(\frac{p}{n}\ \mathbf{K})dn(\frac{p}{n}\ \mathbf{K}) + sn^2(\frac{p}{n}\ \mathbf{K})Z(\frac{p}{n}\ \mathbf{K})}{Z(\frac{p}{n}\ \mathbf{K})}\,.$$

As for $u_m = \sigma_m + i\ \mathbf{K}(\kappa')$ is

$$sn^2(u_m|\kappa) = \frac{1}{\kappa^2 sn^2(\sigma_m|\kappa)}\,, \tag{23}$$

we get the final expression

$$\sigma_m = F\left(\arcsin\left(\frac{1}{\kappa sn(\frac{p}{n}\ \mathbf{K})}\sqrt{\frac{w_m - w_s}{w_m + 1}}\right)|\kappa\right)\,, \tag{24}$$

where $F(\phi|\kappa)$ is the elliptic integral of the first kind. With the substitution (14) we arrive at the standard result (4)

$$Z_{p,q}(u|k) = \frac{(-1)^p}{2}\left[\left(\frac{H(u - \frac{p}{n}\ \mathbf{K}(\kappa))}{H(u + \frac{p}{n}\ \mathbf{K}(\kappa))}\right)^n + \left(\frac{H(u + \frac{p}{n}\ \mathbf{K}(\kappa))}{H(u - \frac{p}{n}\ \mathbf{K}(\kappa))}\right)^n\right]\,. \tag{25}$$

The factor $(-1)^p$ appears here as the generalised Zolotarev polynomial alternates $(p+1)$-times in the interval $(w_p, 1)$ [3]. Using (14), (20) an arbitrary Zolotarev polynomial can be alternatively expressed in terms of the Chebyshev polynomial

$$Z_{p,q}(u|\kappa) = (-1)^p T_n\left(\mathcal{A}_{\frac{p}{n}}(u|\kappa)\right) = \cos n\Phi\,, \tag{26}$$

provided that we define the argument as

$$\mathcal{A}_{\frac{p}{n}}(u|\kappa) = \cos\Phi = \frac{1}{2}\left[\frac{H(u - u_0)}{H(u + u_0)} + \frac{H(u + u_0)}{H(u - u_0)}\right]\,. \tag{27}$$

III. Algebraic Form through the First Principles

From the set of parametric equations (8), (26) and (27) we derive an algebraic form of the simplest Zolotarev polynomial $Z_{p,p}(u|\kappa)$ which is specified by the symmetrical distribution of the zeros in the two disjoint intervals $(-1, -w_p) \cup (w_p, 1)$. Here and in the following, wherever the modulus κ is to be emphasised we use the notation $sn(u|\kappa)$.

In this particular case the variables $u_0 = \frac{1}{2}\ \mathbf{K}(\kappa)$, $sn(\frac{1}{2}\ \mathbf{K}(\kappa)|\kappa) = (1+\kappa')^{-1}$ and

$$w = \frac{sn^2(u|\kappa)(1 - sn^2(\frac{1}{2}\ \mathbf{K}(\kappa)|\kappa)) + sn^2(\frac{1}{2}\ \mathbf{K}(\kappa)|\kappa)}{sn^2(u|\kappa) - sn^2(\frac{1}{2}\ \mathbf{K}(\kappa)|\kappa)}$$

$$= -\frac{1 - (1-\kappa')sn^2(u|\kappa)}{1 - (1+\kappa')sn^2(u|\kappa)} \tag{28}$$

are used. Next, we use the standard notation [4] for the ϑ - functions which assigns for the eta function

$$H(u) = \vartheta_1(v)\,, \tag{29}$$
$$H(u + \mathbf{K}(\kappa)) = \vartheta_2(v)\,,$$

where $v = \frac{\pi}{2\ \mathbf{K}(\kappa)} u$. Then the argument (27) can be written as

$$\mathcal{A}_{\frac{1}{2}}(u|\kappa) = \frac{1}{2}\left[\frac{H(u - \frac{1}{2}\ \mathbf{K}(\kappa))}{H(u + \frac{1}{2}\ \mathbf{K}(\kappa))} + \frac{H(u + \frac{1}{2}\ \mathbf{K}(\kappa))}{H(u - \frac{1}{2}\ \mathbf{K}(\kappa))}\right]$$
$$= \frac{1}{2}\left[\frac{\vartheta_1(v - \frac{\pi}{4})}{\vartheta_2(v - \frac{\pi}{4})} + \frac{\vartheta_2(v - \frac{\pi}{4})}{\vartheta_1(v - \frac{\pi}{4})}\right]$$
$$= \frac{1}{2}\left[\sqrt{\kappa'}\frac{sn(u - \frac{1}{2}\ \mathbf{K}(\kappa)|\kappa)}{cn(u - \frac{1}{2}\ \mathbf{K}(\kappa)|\kappa)} + \frac{1}{\sqrt{\kappa'}}\frac{cn(u - \frac{1}{2}\ \mathbf{K}(\kappa)|\kappa)}{sn(u - \frac{1}{2}\ \mathbf{K}(\kappa)|\kappa)}\right]\,. \tag{30}$$

The pair of equations (28) and (30) already indicates that between the variables w and $\mathcal{A}_{\frac{1}{2}}$ an algebraic relation exists. Using Gauss' transformation for the elliptic functions [4]

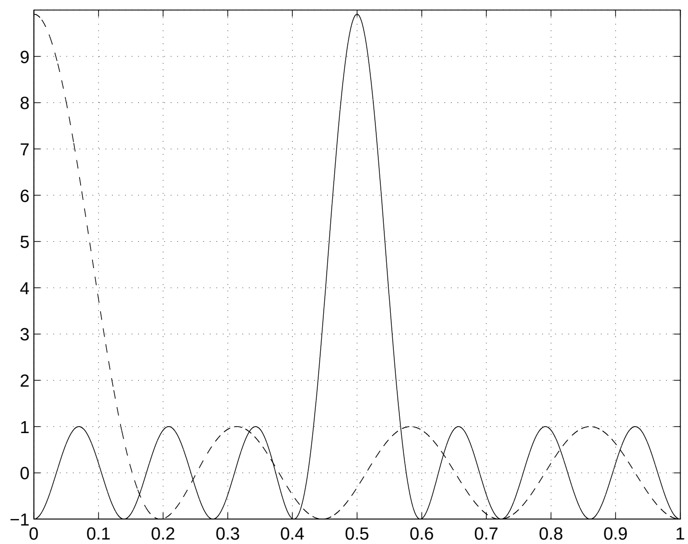

Fig. 2. Symmetrical Zolotarev polynomial $Z_{7,7}(u|0.7575)$ of degree 14, with $\omega_s T = 0.5674\pi$, $\omega_m T = 0.5\pi$ and $\omega_p T = 0.4326\pi$ and corresponding response $W(e^{j\omega})$ of the Chebyshev window function with $\omega_0 T = 0.1347\pi$ plotted versus the normalised frequency - cf. Tab. I.

$$u = \frac{1+k'}{2}(z + \mathbf{K}(k'))\,, \tag{31}$$
$$\kappa = \frac{2\sqrt{k'}}{1+k'}\,,$$

and then letting

$$z = j(y - j\ \mathbf{K}(k')) \tag{32}$$

we obtain the simplified parametric representation

$$w = dn(y|k)\,, \tag{33}$$
$$\mathcal{A}_{\frac{1}{2}} = cn(y|k)\,.$$

Due to this mapping of the independent variable w the edges are related to the new modulus as

$$w_p = -w_s = k'\,. \tag{34}$$

As the standard identity for Jacobi's elliptic functions holds

$$cn^2(y|k) = dn^2(y|k) - k'^2 sn^2(y|k)\,, \tag{35}$$

TABLE I
THE CHEBYSHEV WINDOW FUNCTION AND CORRESPONDING NARROW-BAND FIR FILTER BASED ON THE ZOLOTAREV POLYNOMIAL

Window Function	FIR Narrow-Band Filter
$W(e^{j\omega}) = \sum_{n=-M}^{M} w_M(n)e^{-jn\omega} = T_{2M}\left(\frac{1}{k}cos\frac{\omega}{2}\right)$ $= T_{2M}(sn(u\|k))$ $= T_{2M}\left(\frac{v}{k}\right)$ for $k = cos\frac{\omega_0}{2}$ implicit definition of the window $w_M(n)$	$H(e^{j\omega}) = \sum_{n=0}^{2M} h(n)e^{-jn\omega}$ $= e^{-jM\omega}(-1)^M T_{2M}(cn(u\|k))$ $= e^{-jM\omega}(-1)^M T_M\left(\frac{2w^2-1-k'^2}{1-k'^2}\right)$ the transfer function $H(e^{j\omega})$
$T_{2M}(sn(u\|k))$ $T_{2M}\left(\frac{v}{k}\right) = T_M\left(\frac{2v^2-k^2}{k^2}\right)$ $v^2+w^2=1$	$(-1)^M T_{2M}(cn(u\|k)) = (-1)^M T_M(2cn^2(u\|k)-1)$ $T_M\left(\frac{1+k'^2-2w^2}{1-k'^2}\right) = (-1)^M T_M\left(\frac{2w^2-1-k'^2}{1-k'^2}\right)$ $k^2+k'^2=1$
$w_M(m) =$ $(-1)^M M \sum_{n=m}^{M} \frac{(-1)^n}{M+n}\begin{pmatrix} M+n \\ M-n \end{pmatrix}\begin{pmatrix} 2n \\ n-m \end{pmatrix} k^{-2n}$ for $\mid m \mid \leq M$	$h(M-m) = h(m)$ $(-1)^m M \sum_{n=m}^{M} \frac{(-1)^n}{M+n}\begin{pmatrix} M+n \\ M-n \end{pmatrix}\begin{pmatrix} 2n \\ n-m \end{pmatrix}(1-k'^2)^{-n}$ for m = 0, 1, ...M

the argument $\mathcal{A}^2_{\frac{1}{2}}(y|k)$ and the independent variable w are simply related as

$$2\mathcal{A}^2_{\frac{1}{2}}(y|k) - 1 = \frac{2w^2-1-k'^2}{1-k'^2}\,. \tag{36}$$

Finally, the algebraic form of the symmetrical Zolotarev polynomial $Z_{p,p}(u|\kappa)$ reads

$$Z_{p,p}(w) = (-1)^p T_p\left(\frac{2w^2-1-k'^2}{1-k'^2}\right)\,. \tag{37}$$

This polynomial is equivalent to the implicit definition of the Chebyshev window function [9] - Tab. I.

Though we have demonstrated that replacing of the standard parametric representation of Zolotarev polynomials (3), (4) by an algebraic form is possible, for the general polynomial $Z_{p,q}(u|\kappa)$ this would be a formidable approach. We should have a unified parametrisation of both the argument $\mathcal{A}_{\frac{p}{n}}(u|\kappa)$ - eq.(27) and the independent variable w - eq.(27) in terms of Jacobi's elliptic functions. This means that we should look for expressions in which ratios of $\vartheta-$functions as in eq.(27) are given by elliptic functions. Consequently, it requires general modular transformations of Jacobi's elliptic functions and $\vartheta-$ functions which belong to a rather difficult part of mathematics. But the results will be rewarding. Here, after a modular transformation we have obtained simplified parametric equations of the form (33) which as a result give symmetrically distributed zeros of the Zolotarev polynomial $Z_{p,p}(w)$

$$w_\mu^2 = k'^2 + k^2\cos^2\frac{2\mu-1}{4p}\pi \qquad \mu = 1\ldots p\,. \tag{38}$$

The factorized form of the symmetrical polynomial is then

$$Z_{p,p}(w) = \frac{(-1)^p 2^{2p-1}}{(1-k'^2)^p}\prod_{\mu=1}^{p}(w^2-w_\mu^2)\,. \tag{39}$$

IV. ALGEBRAIC FORM USING LIOUVILLE'S THEOREM

In order to find an algebraic form for Zolotarev's polynomials from the first principles we attempted to express the argument (27) which can be written in terms of theta functions (29)

$$\mathcal{A}_{\frac{p}{n}}(u|\kappa) = \frac{1}{2}\left[\frac{\theta_1(v-v_0)}{\theta_1(v+v_0)} + \frac{\theta_1(v+v_0)}{\theta_1(v-v_0)}\right] \tag{40}$$

through the independent variable w (8) which can be also written in terms of theta functions

$$\begin{aligned} w &= \frac{1}{\kappa^2}\frac{1-dn(u+u_0|\kappa)dn(u-u_0|\kappa)}{sn(u+u_0|\kappa)sn(u-u_0|\kappa)} \qquad (41)\\ &= \frac{\vartheta_4(v+v_0)\vartheta_4(v-v_0) - \kappa'\kappa^2\vartheta_3(v+v_0)\vartheta_3(v-v_0)}{\kappa^3\vartheta_1(v+v_0)\vartheta_1(v-v_0)}\,. \end{aligned}$$

Due to the properties of theta functions the argument $\mathcal{A}_{\frac{p}{n}}(u|\kappa)$ (40) and the independent variable w (41) have the same poles. This consequently means that any polynomial in $\mathcal{A}_{\frac{p}{n}}$ remains a polynomial in the variable w. For a different ratio of zeros $p/q = 1/1, 1/2, 1/3, \ldots$ in the two disjoint intervals $(-1, w_s) \cup (w_p, 1)$ we get the polynomials

$$\begin{aligned} (-1)^p Z_{p,p}(u|\kappa) &= T_{2p}(\mathcal{A}_{\frac{1}{2}}(u|\kappa)) = T_p(T_2(\mathcal{A}_{\frac{1}{2}}))\,, \\ (-1)^p Z_{p,2p}(u|\kappa) &= T_{3p}(\mathcal{A}_{\frac{1}{3}}(u|\kappa)) = T_p(T_3(\mathcal{A}_{\frac{1}{3}}))\,, \\ (-1)^p Z_{p,3p}(u|\kappa) &= T_{4n}(\mathcal{A}_{\frac{1}{4}}(u|\kappa)) = T_p(T_4(\mathcal{A}_{\frac{1}{4}}))\,, \end{aligned}$$

and the search for an algebraic form of a Zolotarev polynomial is reduced to the investigation of an algebraic form of one of the *inner polynomials* $T_2(\mathcal{A}_{\frac{1}{2}}), T_3(\mathcal{A}_{\frac{1}{3}}), \ldots$ only. The inner polynomials are just generators of an arbitrary Zolotarev polynomial. Now we form the ratio of two polynomials $T_n(\mathcal{A}_{\frac{p}{n}}(u|\kappa))/\sum_{\mu=0}^{n} b(\mu)w^\mu$ of the same degree n. This expression is an elliptic function whose numerator and denominator have the same poles and the same zeros. According to Liouville's theorem [12] this ratio must be constant.

From the first principles it is also possible to evaluate the coefficient $b(n)$ accompanying the highest power of w in the general Zolotarev polynomial. If we use the representation (26) it turns out that the limit $w \to \infty$ is equivalent to the limit $v \to v_0$ and then

$$\begin{aligned} b(n) = \lim_{v\to v_0} \frac{T_n(\mathcal{A}_{\frac{1}{n}})}{w^n} &= \frac{1}{2}\left\{\frac{\vartheta_4(2v_0)}{\vartheta_4(0)} + \frac{\vartheta_3(2v_0)}{\vartheta_3(0)}\right\}^n \quad (42) \\ &= \frac{1}{2}\left\{\frac{\vartheta_4(2v_0)}{\vartheta_4(0)}[1 + dn(2u_0|\kappa)]\right\}^n, \end{aligned}$$

where $2v_0 = \pi/n$. This expression again confirms that for the general Zolotarev polynomial the evaluation of the coefficients is closely related to modular transformations of $\vartheta-$functions. We have employed Liouville's theorem and the values of the highest coefficients in the evaluation of an algebraic form of the polynomials $T_2(\mathcal{A}_{\frac{1}{2}}), T_3(\mathcal{A}_{\frac{1}{3}})$ and $T_4(\mathcal{A}_{\frac{1}{4}})$. We now show how the third order Zolotarev polynomial is developed. First assume the identity

$$\frac{4\mathcal{A}^3_{\frac{1}{3}}(u|\kappa)) - 3\mathcal{A}_{\frac{1}{3}}(u|\kappa))}{b(3)w^3 + b(2)w^2 + b(1)w + b(0)} = 1 \quad (43)$$

and then by investigating the behaviour of both polynomials - cf. Fig. 3 - in specific points we write the set of algebraic equations

$$\begin{aligned} &\text{at} \quad w = -1 && 4\mathcal{A}^3_{\frac{1}{3}} - 3\mathcal{A}_{\frac{1}{3}} = -1\,, && (44) \\ &&& -b(3) + b(2) - b(1) + b(0) = -1\,, \\ &\text{at} \quad w = w_s && 4\mathcal{A}^3_{\frac{1}{3}} - 3\mathcal{A}_{\frac{1}{3}} = -1\,, && (45) \\ &&& b(3)w_s^3 + b(2)w_s^2 + b(1)w_s + b(0) = -1\,, \\ &\text{at} \quad w = w_p && 4\mathcal{A}^3_{\frac{1}{3}} - 3\mathcal{A}_{\frac{1}{3}} = -1\,, && (46) \\ &&& b(3)w_p^3 + b(2)w_p^2 + b(1)w_p + b(0) = -1\,, \\ &\text{at} \quad w = 1 && 4\mathcal{A}^3_{\frac{1}{3}} - 3\mathcal{A}_{\frac{1}{3}} = 1\,, && (47) \\ &&& b(3) + b(2) + b(1) + b(0) = 1\,. \end{aligned}$$

Observing that $b(2) + b(0) = 0$ the set of equations (41) - (44) can be written in the matrix form

$$\begin{bmatrix} 1 & 0 & 1 \\ w_s^3 & w_s^2 - 1 & w_s \\ w_p^3 & w_p^2 - 1 & w_p \end{bmatrix} \begin{bmatrix} b(3) \\ b(2) \\ b(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad (48)$$

which can be easily inverted giving the solution in terms of $w_s = 1 - 2sn^2\left(\frac{\mathbf{K}}{3}\right)$ and $w_p = 2cd^2\left(\frac{\mathbf{K}}{3}\right) - 1$

$$\begin{aligned} b(3)\Delta &= (w_s - w_p)(1 + w_p w_s) - (1 - w_p^2) - (1 - w_s^2)\,, \\ b(2)\Delta &= w_p w_s(w_s^2 - w_p^2) - w_p(1 - w_p^2) - w_s(1 - w_s^2)\,, \\ b(1)\Delta &= w_p^3 - w_s^3 + w_p^2 w_s^2(w_s - w_p) + 2 - w_p^2 - w_s^2\,, \\ b(0)\Delta &= -w_p w_s(w_s^2 - w_p^2) + w_p(1 - w_p^2) + w_s(1 - w_s^2)\,. \end{aligned}$$

where $\Delta = (w_s - w_p)(1 - w_p^2)(1 - w_s^2)$ is the determinant of the matrix in eq. (48). It is worth noticing that $Z_{2,1}(w) = Z_{1,2}(-w)$.

In the case of the fourth order Zolotarev polynomial we investigate the behaviour of the expression

$$\frac{8\mathcal{A}^4_{\frac{1}{4}}(u|\kappa)) - 8\mathcal{A}^2_{\frac{1}{4}}(u|\kappa)) + 1}{b(4)w^4 + b(3)w^3 + b(2)w^2 + b(1)w + b(0)} = 1\,. \quad (49)$$

As in the previous case we obtain the set of algebraic equations

$$\begin{aligned} & 8\mathcal{A}^4_{\frac{1}{4}} - 8\mathcal{A}^2_{\frac{1}{4}} + 1 = 1\,, && (50) \\ & b(4) - b(3) + b(2) - b(1) + b(0) = -1\,, \\ & 8\mathcal{A}^4_{\frac{1}{4}} - 8\mathcal{A}^2_{\frac{1}{4}} + 1 = -1\,, && (51) \\ & b(4)w_s^4 + b(3)w_s^3 + b(2)w_s^2 + b(1)w_s + b(0) = -1\,, \\ & 8\mathcal{A}^4_{\frac{1}{4}} - 8\mathcal{A}^2_{\frac{1}{4}} + 1 = -1\,, && (52) \\ & b(4)w_p^4 + b(3)w_p^3 + b(2)w_p^2 + b(1)w_p + b(0) = -1\,, \\ & 8\mathcal{A}^4_{\frac{1}{4}} - 8\mathcal{A}^2_{\frac{1}{4}} + 1 = 1\,, && (53) \\ & b(4) + b(3) + b(2) + b(1) + b(0) = 1\,. \end{aligned}$$

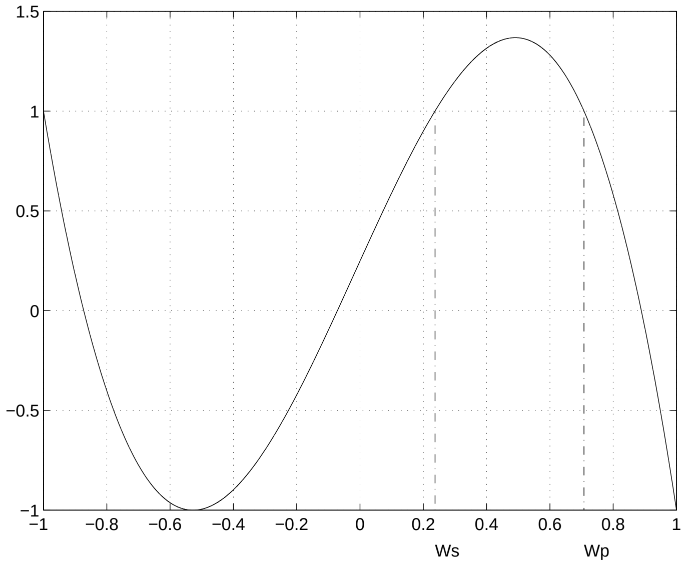

Fig. 3. Polynomial $T_3(\mathcal{A}_{\frac{1}{3}}(u|\kappa))$, with $\kappa = 0.85$ with $w_s = 0.2369$ and $w_p = 0.7077$.

TABLE II

COEFFICIENTS OF THE LOWEST ORDER ZOLOTAREV POLYNOMIALS $Z_{1,n-1}(w) = b(n)\sum_{m=0}^{n}\beta(m)w^m$

normalised coefficients	n=2	n=3	n=4
$\beta(0)$	$-1+2sn^2\left(\frac{\mathbf{K}}{2}\right)cn^2\left(\frac{\mathbf{K}}{2}\right)$	$-\left(1-2sn\left(\frac{\mathbf{K}}{3}\right)\right)^2$	$-\beta-4\frac{sn\left(\frac{\mathbf{K}}{2}\right)cn\left(\frac{\mathbf{K}}{2}\right)dn\left(\frac{\mathbf{K}}{2}\right)}{\left(1+dn\left(\frac{\mathbf{K}}{2}\right)\right)^4}$
$\beta(1)$	0	$4sn^2\left(\frac{\mathbf{K}}{3}\right)cn^2\left(\frac{2\mathbf{K}}{3}\right)-1$	$-2sn\left(\frac{\mathbf{K}}{2}\right)\frac{\left(1-dn\left(\frac{\mathbf{K}}{2}\right)\right)^2}{1+dn\left(\frac{\mathbf{K}}{2}\right)}$
$\beta(2)$	1	$\left(1-2sn\left(\frac{\mathbf{K}}{3}\right)\right)^2$	$-1+\beta+8\frac{sn\left(\frac{\mathbf{K}}{2}\right)cn\left(\frac{\mathbf{K}}{2}\right)dn\left(\frac{\mathbf{K}}{2}\right)}{\left(1+dn\left(\frac{\mathbf{K}}{2}\right)\right)^4}$
$\beta(3)$	-	1	$2sn\left(\frac{\mathbf{K}}{2}\right)\frac{\left(1-dn\left(\frac{\mathbf{K}}{2}\right)\right)^2}{1+dn\left(\frac{\mathbf{K}}{2}\right)}$
$\beta(4)$	-	-	1
$b(n)$	$\frac{1}{2sn^2\left(\frac{\mathbf{K}}{2}\right)cn^2\left(\frac{\mathbf{K}}{2}\right)}$	$\frac{1}{4sn^2\left(\frac{\mathbf{K}}{3}\right)cn^2\left(\frac{2\mathbf{K}}{3}\right)}$	$\frac{sn\left(\frac{\mathbf{K}}{2}\right)cn\left(\frac{\mathbf{K}}{2}\right)}{4sn^2\left(\frac{\mathbf{K}}{4}\right)sn^2\left(\frac{3\mathbf{K}}{4}\right)cn^2\left(\frac{\mathbf{K}}{4}\right)cn^2\left(\frac{3\mathbf{K}}{4}\right)}$
β	$\frac{dn^2(\frac{1}{2}\mathbf{K})}{sn^2(\frac{1}{2}\mathbf{K})dn^2(\frac{1}{2}\mathbf{K})}\frac{1-2sn(\frac{1}{2}\mathbf{K})cn(\frac{1}{2}\mathbf{K})(sn(\frac{1}{2}\mathbf{K})+cn(\frac{1}{2}\mathbf{K}))^2}{(1+dn(\frac{1}{2}\mathbf{K}))^4}$		

In order to achieve completeness of the set of equations for 5 unknown coefficients we have to use the identity (42) which for $n = 4$ gives

$$b(4) = \frac{1}{2}\frac{\vartheta^4(\pi/4)}{\vartheta^4(0)}(1+dn(\frac{1}{2}\mathbf{K}))^4 = \frac{(1+\kappa')(1+\sqrt{\kappa'})^4}{4\sqrt{\kappa'}\kappa'} \tag{54}$$

Substituting $b(4)$ from eq.(54) and considering $b(3)+b(1) = 0$ the set of equations (50) - (53) can be reduced to the matrix form

$$\begin{bmatrix} 0 & 1 & \\ w_s^3 - w_s & w_s^2 & 1 \\ w_p^3 - w_p & w_p^2 & 1 \end{bmatrix}\begin{bmatrix} b(3) \\ b(2) \\ b(0)\end{bmatrix} = \begin{bmatrix} 1-b(4) \\ -1-b(4)w_s^4 \\ -1-b(4)w_p^4\end{bmatrix}. \tag{55}$$

By inversion of (55) we can express the solution in the form

$$b(3) = \frac{2(w_s+w_p)}{(1-w_p^2)(1-w_s^2)} - b(4)(w_s+w_p)\,, \tag{56}$$

$$b(2) = 2 - b(4)(1-w_pw_s) - \frac{2w_sw_p(1+w_sw_p)}{(1-w_p^2)(1-w_s^2)}, \tag{57}$$

$$b(1) = -\frac{2(w_s+w_p)}{(1-w_p^2)(1-w_s^2)} + b(4)(w_s+w_p)\,, \tag{58}$$

$$b(0) = 1 - b(4)w_pw_s + \frac{2w_sw_p(1+w_sw_p)}{(1-w_p^2)(1-w_s^2)}\,. \tag{59}$$

Note that the polynomials already computed cover also $Z_{3,1}(w) = Z_{1,3}(-w)$ and $Z_{2,2}(w) = T_2(Z_{1,1}(w))$.

The coefficients of the lowest order Zolotarev polynomials are recomputed in terms of Jacobi's elliptic functions and summarised in Tab. II.

The bandpass FIR filter of length 41 designed in [3] and reproduced here in Fig.4 is based on Zolotarev polynomial $Z_{5,15}(u|0.77029)$ of degree 20

$$Z_{5,15}(u|0.77029) = T_5(Z_{1,3}(u|0.77029)) \equiv T_5(Z_{1,3}(w))\,. \tag{60}$$

The impulse response coefficients

$$T_5(Z_{1,3}(w)) = \sum_{m=0}^{20} a(m)T_m(w) \tag{61}$$

can be evaluated by spectral transformation or by FFT transform - cf. Table III. It is one disadvantage of designing FIR filters with inner polynomials.

V. LINEAR DIFFERENTIAL EQUATION AND RECURSIVE EVALUATION OF COEFFICIENTS

The approximation equation (7) is nonlinear and cannot be easily used to remove the parametrisation and find the algebraic form of a Zolotarev polynomial as

$$Z_{p,q}(w) = \sum_{m=0}^{n} b(m)w^m\,. \tag{62}$$

TABLE III
THE IMPULSE RESPONSE COEFFICIENTS OF $T_5(Z_{1,3}(u|0.77029))$

n		$h(n)$
0	40	-0.643099
1	39	-0.226762
2	38	-0.035981
3	37	0.223379
4	36	0.387465
5	35	0.325961
6	34	0.039811
7	33	-0.317256
8	32	-0.520533
9	31	-0.416778
10	30	-0.036271
11	29	0.406232
12	28	0.634285
13	27	0.485255
14	26	0.025390
15	25	-0.476835
16	24	-0.710942
17	23	-0.520577
18	22	-0.009122
19	21	0.517382
20		0.737995

Consequently, we take the first derivative of eq. (7) which after some algebra leads to the second order linear differential equation

$$g_2(w)[(1-w^2)\frac{d^2f}{dw^2}-w\frac{df}{dw}]-(1-w^2)g_1(w)\frac{df}{dw}+g_0(w)f=0\,, \tag{63}$$

where

$$\begin{aligned} g_2(w) &= (w-w_p)(w-w_s)(w-w_m)\,, \\ g_1(w) &= (w-w_p)(w-w_s)-(w-w_m)(w-\frac{w_p+w_s}{2})\,, \\ g_0(w) &= n^2(w-w_m)^3\,. \end{aligned} \tag{64}$$

This differential equation being linear is suitable for the solution with the power series. By substituting

$$\begin{aligned} f(w) &= \sum_{m=0}^{n} b(m)w^m\,, \\ f'(w) &= \sum_{m=0}^{n-1}(m+1)b(m+1)w^m\,, \\ f''(w) &= \sum_{m=0}^{n-2}(m+2)(m+1)b(m+2)w^m\,. \end{aligned} \tag{65}$$

in the linear differential equation (63) and comparing the coefficients with the same power of w we obtain a set of recursive formulae concisely summarised in Tab. IV. Note that the recursion is a convolution with time varying coefficients $\{d(1),d(2),d(3),d(4),d(5),d(6)\}$

$$d(6)b(m+3-6)=\sum_{\mu=1}^{5}d(\mu)b(m+3-\mu)\,;\; m=n+2,\ldots,3 \tag{66}$$

which in each consecutive step predicts a new coefficient of the Zolotarev polynomial. The nonzero initial value is taken $b(n)=1$ then all values $b(m)$ for $m=n-1,\ldots,2,1,0$ are obtained and finally re-normalisation is performed using the value of the Zolotarev polynomial at $w=1$.

The algorithm gives not only an efficient code for the evaluation of the Zolotarev polynomials but provides a purely analytical view on the coefficients. By analytic iteration we can obtain general relations among the coefficients as

$$\frac{b(n-1)}{nb(n)} = w_m-w_q\,, \tag{67}$$

$$\frac{4b(n-2)}{nb(n)} = 3w_m(w_m-w_q)+(2n-3)(w_m-w_q)^2 +w_pw_s-w_mw_q-1\,. \tag{68}$$

It is worth to note that coefficient $b(n-1)$ is related to σ from eq.(1) through the transformation (6)

$$-\frac{b(n-1)}{b(n)}=\sigma cn^2(u_0|\kappa)-sn^2(u_0|\kappa)\,, \tag{69}$$

which gives σ used by N. I. Achieser [2]

$$\sigma=\frac{2sn(u_0|\kappa)}{cn(u_0|\kappa)dn(u_0|\kappa)}\left[\frac{1}{sn(2u_0|\kappa)}-Z(u_0)\right]-1\,. \tag{70}$$

VI. CHEBYSHEV EXPANSION OF ZOLOTAREV POLYNOMIALS

We wrote the linear differential equation purposely in the form (63) which suggests to use Chebyshev polynomials of the first kind $T_m(w)$ in the expansion of Zolotarev polynomials

$$Z_{p,q}(w)=\sum_{m=0}^{n}a(m)T_m(w)\,. \tag{71}$$

Indeed, using the differential properties of Chebyshev polynomials for an expansion

$$f(w)=\sum_{m=0}^{n}a(m)T_m(w)\,, \tag{72}$$

we can write

$$(1-w^2)\frac{d^2f}{dw^2}-w\frac{df}{dw}=-\sum_{m=0}^{n}m^2a(m)T_m(w)\,, \tag{73}$$

$$(1-w^2)\frac{df}{dw}=\sum_{m=0}^{n}ma(m)[T_{m-1}(w)-wT_m(w)]\,. \tag{74}$$

The linear differential equation (63) has then the form

$$\begin{aligned} &-\sum_{m=0}^{n}m^2a(m)g_2(w)T_m(w) \\ &\quad-\sum_{m=0}^{n}ma(m)g_1(w)[T_{m-1}(w)-wT_m(w)] \\ &\quad\quad+\sum_{m=0}^{n}a(m)g_0(w)T_m(w)=0\,. \end{aligned} \tag{75}$$

In order to compare the coefficients associated with the Chebyshev polynomials $T_m(w)$ of the same order we have to

TABLE IV
BACKWARD RECURSIVE ALGORITHM FOR EVALUATION OF ZOLOTAREV POLYNOMIALS $Z_{p,q}(w) = \sum_{m=0}^{n} b(m) w^m$

given	
	p, q
initialisation	
	$n = p + q$
eq. (9)	$w_p = 2\,cd^2(u_0\|\kappa) - 1$
eq. (10)	$w_s = 2\,cn^2(u_0\|\kappa) - 1$
	$w_q = \frac{w_p + w_s}{2}$
eq. (19)	$w_m = w_s + 2\frac{sn(u_0)cn(u_0)}{dn(u_0)} Z(u_0)$
	$\beta(n) = 1$
	$\beta(n+1) = \beta(n+2) = \beta(n+3) = \beta(n+4) = 0$
body	
(*for*	$m = n+2$ *to* 3)
	$d(1) = (m+2)(m+1) w_p w_s w_m$
	$d(2) = -(m+1)(m-1) w_p w_s - (m+1)(2m+1) w_m w_q$
	$d(3) = w_m (n^2 w_m^2 - m^2 w_p w_s) + m^2 (w_m - w_q) + 3m(m-1) w_q$
	$d(4) = (m-1)(m-2)(w_p w_s - w_m w_q - 1) - 3 w_m (n^2 w_m - (m-1)^2 w_q)$
	$d(5) = (2m-5)(m-2)(w_m - w_q) + 3 w_m [n^2 - (m-2)^2]$
	$d(6) = n^2 - (m-3)^2$
	$\beta(m-3) = \frac{1}{d(6)} \sum_{\mu=1}^{5} d(\mu) \beta(m+3-\mu)$
(*end*	*loop on* m)
normalisation	
	$s(n) = \sum_{m=0}^{n} \beta(m)$
(*for*	$m = 0$ *to* n)
	$b(m) = (-1)^p \frac{\beta(m)}{s(n)}$
(*end*	*loop on* m)

remove all the multiplications $w^k T_m(w)$. Using the recursive formula for Chebyshev polynomials

$$
\begin{aligned}
(2w)^1 T_m(w) &= T_{m-1}(w) + T_{m+1}(w)\,,\\
(2w)^2 T_m(w) &= T_{m-2}(w) + 2T_m(w) + T_{m+2}(w)\,,\\
(2w)^3 T_m(w) &= T_{m-3}(w) + 3T_{m-1}(w)\\
&\quad + 3T_{m+1}(w) + T_{m+3}(w)\,.
\end{aligned} \tag{76}
$$

and rearranging the summation in equation (75) we arrive at a recursive evaluation of the coefficients $a(m)$. It is again a convolution with time varying coefficients $\{c(1), c(2), c(3), c(4), c(5), c(6), c(7)\}$

$$
c(7) a(m+4-7) = \sum_{\mu=1}^{6} c(\mu) a(m+4-\mu)\,;\; m = n+2, \ldots, 3 \tag{77}
$$

The first nonzero value is taken $a(n) = 1$, then all values $a(m)$ for $m = n-1, \ldots, 2, 1, 0$ are obtained and finally renormalised. The algorithm is concisely summarised in Tab. V. Our algorithm gives directly the impulse response coefficients $h(m)$

$$
\begin{aligned}
a(0) &= h(M)\,,\\
a(m) &= 2h(M-m)
\end{aligned} \tag{78}
$$

of a narrow band FIR filter of length $N = 2M + 1 = 2(p + q) + 1$. Its transfer function is given as

$$
\begin{aligned}
H(z) &= \sum_{\nu=0}^{N-1} h(\nu)\, z^{-\nu} = z^{-M}\left[a(0) + \sum_{m=1}^{M} a(m)\, T_m(w)\right]\\
&= z^{-M} Z_{p,q}(w)\,.
\end{aligned} \tag{79}
$$

Solely from the numerical point of view the latter algorithm is rather advantageous as it offers a lower range of coefficients which affects the rounding error.

VII. FIR FILTERS APPLICATIONS

Both recursive algorithms for the coefficients of Zolotarev's polynomials provide fundamental tools for the design of several types of FIR filters.

First, we consider the design of a bandpass filter. The proposed procedure for designing optimal bandpass FIR filters

TABLE V
BACKWARD RECURSIVE ALGORITHM FOR EVALUATION OF ZOLOTAREV POLYNOMIALS $Z_{p,q}(w)=\sum_{m=0}^{n} a(m)T_m(w)$

given	
	p, q
initialisation	
	$n = p + q$
eq. (9)	$w_p = 2\,cd^2(u_0\|\kappa) - 1$
eq. (10)	$w_s = 2\,cn^2(u_0\|\kappa) - 1$
	$w_q = \frac{w_p + w_s}{2}$
eq. (19)	$w_m = w_s + 2\frac{sn(u_0)cn(u_0)}{dn(u_0)}Z(u_0)$
	$\alpha(n) = 1$
	$\alpha(n+1) = \alpha(n+2) = \alpha(n+3) = \alpha(n+4) = \alpha(n+5) = 0$
body	
(for	$m = n+2$ *to* $3)$
	$8c(1) = n^2 - (m+3)^2$
	$4c(2) = (2m+5)(m+2)(w_m - w_q) + 3w_m[n^2 - (m+2)^2]$
	$2c(3) = \frac{3}{4}[n^2 - (m+1)^2] + 3w_m[n^2 w_m - (m+1)^2 w_q] - (m+1)(m+2)(w_p w_s - w_m w_q)$
	$c(4) = \frac{3}{2}(n^2 - m^2) + m^2(w_m - w_q) + w_m(n^2 w_m^2 - m^2 w_p w_s)$
	$2c(5) = \frac{3}{4}[n^2 - (m-1)^2] + 3w_m[n^2 w_m - (m-1)^2 w_q] - (m-1)(m-2)(w_p w_s - w_m w_q)$
	$4c(6) = (2m-5)(m-2)(w_m - w_q) + 3w_m[n^2 - (m-2)^2]$
	$8c(7) = n^2 - (m-3)^2$
	$\alpha(m-3) = \frac{1}{c(7)}\sum_{\mu=1}^{6} c(\mu)\alpha(m+4-\mu)$
(end	*loop on* $m)$
normalisation	
	$s(n) = \frac{\alpha(0)}{2} + \sum_{m=1}^{n}\alpha(m)$
	$a(0) = (-1)^p\frac{\alpha(0)}{2s(n)}$
(for	$m = 1$ *to* $n)$
	$a(m) = (-1)^p\frac{\alpha(m)}{s(n)}$
(end	*loop on* $m)$

is a simplified version of that given by X. Chen and T. W. Parks [3]. It is free of the transformation from $(-1,1)\cup(\alpha,\beta)$ to the digital domain $(-1,w_s)\cup(w_p,1)$ and it does not require any FFT algorithm. Auxiliary parameters φ_p, φ_s related to the partition of the quarter-period **K** are introduced

$$\frac{p}{n}\mathbf{K}(\kappa) + \frac{q}{n}\mathbf{K}(\kappa) = F(\varphi_s|\kappa) + F(\varphi_p|\kappa) = \mathbf{K}(\kappa)\,, \qquad (80)$$

where

$$\frac{p}{n}\mathbf{K}(\kappa) = F(\varphi_s|\kappa)\,, \qquad \frac{q}{n}\mathbf{K}(\kappa) = F(\varphi_p|\kappa) \qquad (81)$$

are incomplete elliptic integrals of the first kind. The new auxiliary parameters reduce the computation of the elliptic function to the standard trigonometric functions as

$$\sin\varphi_s = sn\left(\frac{p}{n}\mathbf{K}(\kappa)\right)\,, \qquad (82)$$

$$\sin\varphi_p = sn\left(\frac{q}{n}\mathbf{K}(\kappa)\right)\,. \qquad (83)$$

The procedure is as follows.

1) Specify the desired stopband edges $\omega_p < \omega_s$ and stopband ripple δ.
2) Evaluate the modulus of Jacobi's elliptic functions κ for $\varphi_s = \frac{\omega_s T}{2}$ and $\varphi_p = \frac{\pi - \omega_p T}{2}$

$$\kappa' = \frac{1}{\tan(\varphi_s)\tan(\varphi_p)}\,. \qquad (84)$$

3) Compute the minimum degree n needed to satisfy the attenuation of the stopband ripples. This requires the simultaneous solution of partition equation (80) and the degree equation

$$n = \frac{\ln(y_m + \sqrt{y_m^2 - 1})}{2\sigma_m Z(\frac{p}{n}\mathbf{K}(\kappa)|\kappa) - 2\Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\kappa)}\,, \qquad (85)$$

where

$$y_m = 10^{0.05\delta} \qquad (86)$$

TABLE VI
COMPARISON OF DYNAMIC RANGE OF COEFFICIENTS FOR REPRESENTATION OF POLYNOMIAL $Z_{3,6}(u|0.682)$ FROM FIG. 6

m	$a(m)$	$b(m)$
0	0.098598	-0.1674
1	0.097937	-9.2167
2	-0.098642	6.6731
3	-0.193401	132.4135
4	-0.093506	-23.7022
5	0.095518	-477.1399
6	0.182318	28.5587
7	0.085744	630.9074
8	-0.088768	-11.3623
9	-1.085798	-277.9644

corresponds to the maximum of the Zolotarev polynomial at the point w_m. The degree equation follows from eqs. (16), (17), (24) and (27). Note that it is a true degree equation as all variables $\sigma_m, \Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\kappa), Z(\frac{p}{n}\mathbf{K}(\kappa)|\kappa)$ are due to the partition equation (80) explicitly independent of n.

4) Use eq. (81) to determine integer values of p and q.
5) Compute the actual values of ω_p, ω_s and ω_m as

$$\begin{aligned} w_p = \cos\omega_p T &= 2sn^2\left(\frac{q}{n}\mathbf{K}(\kappa)\right) - 1\,, \\ w_s = \cos\omega_s T &= 1 - 2sn^2\left(\frac{p}{n}\mathbf{K}(\kappa)\right)\,, \\ w_m = \cos\omega_m T &= w_s + 2\,\frac{dn(u_0)}{sn(u_0)cn(u_0)} Z(u_0)\,. \end{aligned}$$

6) For integer values p and q carry out the algorithm giving the impulse response coefficients

$$\begin{aligned} a(0) &= h(M)\,, \\ a(m) &= 2h(M-m)\,. \end{aligned} \quad (87)$$

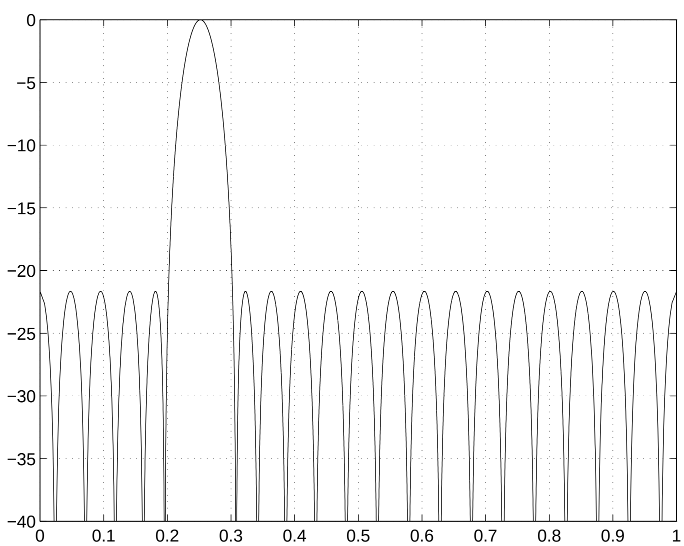

Fig. 4. Bandpass filter of length 41 based on Zolotarev polynomial $Z_{5,15}(u|0.77029)$ of degree 20, with $\omega_s T = 0.3023\pi$, $\omega_m T = 0.2520\pi$ and $\omega_p T = 0.2017\pi$. Frequency response $H(e^{j\omega})$ is plotted versus the normalised frequency. The ripples in the stopbands are less than -21.65 dB.

The FIR bandpass filters obtained are maximum ripple filters so that the only available stopband edges are discretised by eq.(80). This is naturally different from the filters designed by the Parks-McClellan program where band edges are adjusted by one or more extra zeros which are off the unit circle [6]. Zolotarev's polynomials have no other zeros than those on the unit circle and therefore they satisfy only the band-edge requirements constrained by eq.(80). Strict approximation requirements usually give such discrete values for the positions of stopband/passband edges.

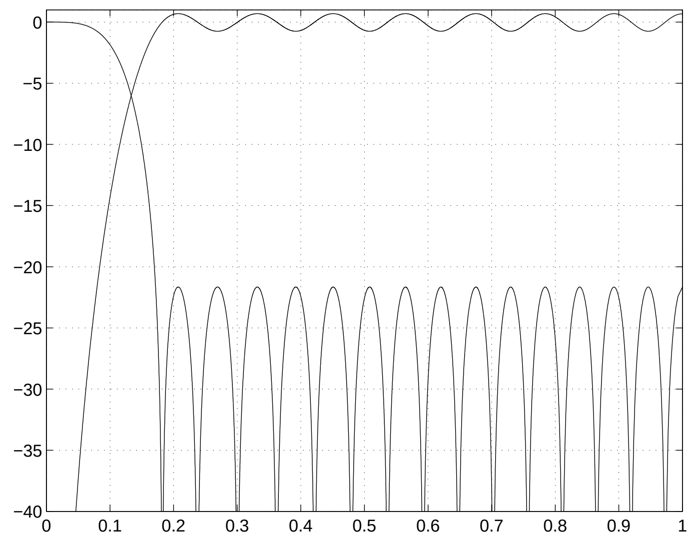

Fig. 5. Complementary FIR filter pair with $\bar{\omega}_s T = 0.1717\pi$ transformed from the bandpass in Fig. 4. Frequency response $H(e^{j\omega})$ is plotted versus the normalised frequency.

Second, we design a complementary pair of FIR filters based on a Zolotarev polynomial

$$Z_{p,q}(w) = \sum_{m=0}^{n} a(m) T_m(w) \quad (88)$$

by linear transformation

$$w = \frac{1+w_m}{2}\bar{w} - \frac{1-w_m}{2}\,. \quad (89)$$

Third, we introduce the design of almost equiripple double-notch FIR filters. The procedure is based on the observation that the odd part of a Zolotarev polynomial has two extra lobes for which

$$\left|\frac{1}{2}\left(Z_{p,q}(w) - Z_{p,q}(-w)\right)\right| > 1\,, \quad (90)$$

which are of the same magnitude. Substituting the odd part of a Zolotarev polynomial in a Chebyshev polynomial we generate the transfer function of a double-notch FIR filter using

$$Q(w) = T_r(Z_{p,q}(w))\,. \quad (91)$$

The transfer function of a double-notch FIR filter of length $2M+1 = r(p+q)+1$ is then

$$H(z) = z^{-M}\left(1 - \frac{Q(w)}{Q(w_{max})}\right)\,. \quad (92)$$

Note that the maximum occurs at w_{max} which is slightly different from the value w_m which belongs to the maximum of a Zolotarev polynomial. In the example in Fig. 6 the differences are as follows

$$w_{max} = \pm 0.5018 \qquad w_{m\pm} = \pm 0.4977\,. \tag{93}$$

The ripples in the passband are not exactly equal but they fall within the limit of ripples of an optimal single notch filter. Such FIR filters will play an important role in filtering of the sinusoidal interference harmonics.

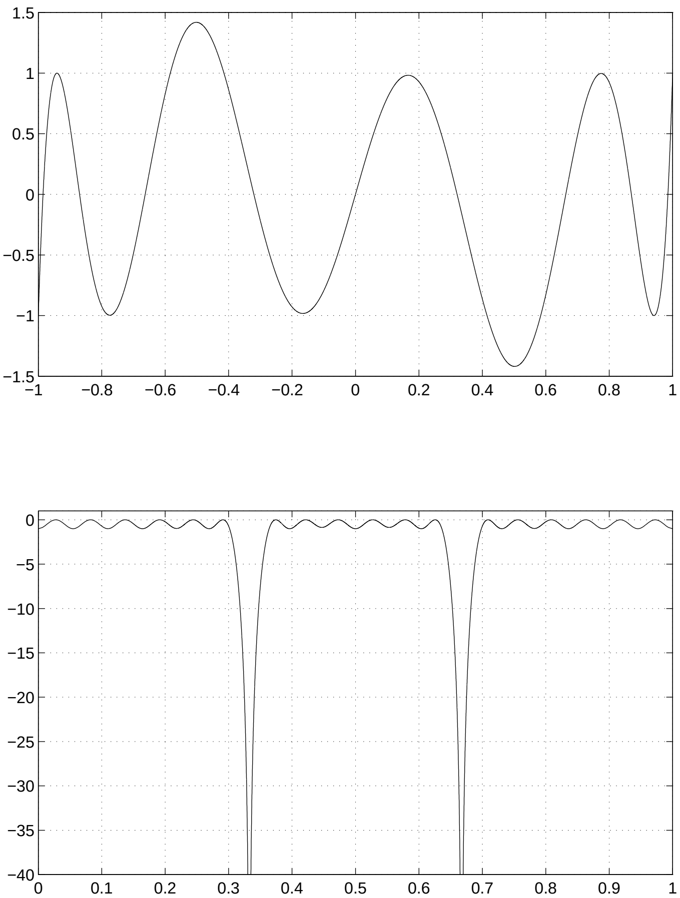

Fig. 6. Odd part of Zolotarev polynomial $Z_{3,6}(u|0.682)$ of degree 9, with $\omega_s T = 0.3771\pi$, $\omega_m T = 0.3342\pi$ and $\omega_p T = 0.2912\pi$ and the frequency response of the corresponding FIR double-notch filter generated by $T_4(x)$. Frequency response $H(e^{j\omega})$ with notch frequencies specified by $\omega_{0+}T = 0.3327\pi$ and $\omega_{0-}T = 0.6673\pi$ is plotted versus the normalised frequency. The ripples in the passband are less than 1 dB.

VIII. CONCLUDING REMARKS

We have presented a purely algebraic solution for Zolotarev polynomials which completely replaces so far used parametric solutions for these polynomials. The recursive algorithms we have derived are well suited for the design of optimal narrow-band FIR filters. The second algorithm leads directly to the impulse response coefficients of a narrow-band filter. The core of the solution is seen in linear differential equation for a general Zolotarev polynomial which is to our knowledge a new concept in approximation problems. The linear differential equation then yields solutions for both representations (62) and (71). Apart from usual FIR filters we have proposed the design of almost equiripple double-notch FIR filters. The algorithms give not only an efficient code for evaluation of Zolotarev polynomials but provide a purely analytical view on the coefficients.

There are more mathematical problems to be solved such as the problem of distribution of the zeros or the orthogonality of Zolotarev polynomials. The solutions of these problems will presumably affect several signal processing algorithms.

IX. APPENDIX I - EVALUATION OF MAXIMUM OF ZOLOTAREV POLYNOMIALS

For Jacobi's zeta function the addition theorem holds

$$Z(u|\kappa)+Z(v|\kappa)-Z(u+v|\kappa) = \kappa^2 sn(u|\kappa)sn(v|\kappa)sn(u+v|\kappa).$$

The addition theorem relates the single periodic function $Z(u)$ to the double periodic Jacobi's elliptic function $sn(u)$. This is the reason why there is no algebraic relation which connects $Z(u)$ with $sn(u), cn(u)$ and $dn(u)$ [5] and why this formula is often called quasi-addition theorem [4]. Consequently the numerical evaluation of $Z(u)$ is usually performed using an arithmetic-geometric mean procedure [1] omitting the addition theorem. In our application the argument u is attributed to the specific discrete values of the half-period and the zeta function is not necessarily evaluated independently of Jacobi's elliptic functions. For Jacobi's zeta function $Z(u|\kappa)$ of a discrete argument $u_m = \dfrac{m}{n}\,\mathbf{K}(\kappa)$ we have used the addition theorem to prove the algebraic formula [10]

$$[Z] = \kappa^2 \frac{sn\left(\frac{1}{n}\,\mathbf{K}\right)}{n}\left(\,\mathbf{A} - n\mathbf{B}\right)[S]\;, \tag{94}$$

where the abbreviated notation for vectors is introduced

$$[Z] \;=\; \begin{bmatrix} Z\left(\dfrac{n-1}{n}\,\mathbf{K}\right) \\ \vdots \\ Z\left(\dfrac{2}{n}\,\mathbf{K}\right) \\ Z\left(\dfrac{1}{n}\,\mathbf{K}\right) \end{bmatrix} \tag{95}$$

$$[S] \;=\; \begin{bmatrix} sn\left(\dfrac{n-1}{n}\,\mathbf{K}\right) sn\left(\dfrac{n}{n}\,\mathbf{K}\right) \\ \vdots \\ sn\left(\dfrac{2}{n}\,\mathbf{K}\right) sn\left(\dfrac{3}{n}\,\mathbf{K}\right) \\ sn\left(\dfrac{1}{n}\,\mathbf{K}\right) sn\left(\dfrac{2}{n}\,\mathbf{K}\right) \end{bmatrix}. \tag{96}$$

In this equation (94) the upper triangular matrix **U** of units and lower triangular matrix **L** of units, are used in the following sense

$$\mathbf{B} = \mathbf{U} - \mathbf{1}, \tag{97}$$
$$\mathbf{A} = (n\,\mathbf{1} - \mathbf{L})(\mathbf{L} + \mathbf{U} - \mathbf{1}). \tag{98}$$

Note that both $\mathbf{A}$ and $\mathbf{B}$ are singular matrices. The equation (94) can be also written in a scalar form

$$Z\left(\frac{p}{n}\,\mathbf{K}\right) = \frac{\kappa^2 sn\left(\frac{1}{n}\,\mathbf{K}\right)}{n} \times \times\left\{p\sum_{m=1}^{n-1} sn\left(\frac{m}{n}\,\mathbf{K}\right) sn\left(\frac{m+1}{n}\,\mathbf{K}\right) -n\sum_{m=1}^{p-1} sn\left(\frac{m}{n}\,\mathbf{K}\right) sn\left(\frac{m+1}{n}\,\mathbf{K}\right)\right\}. \tag{99}$$

The algebraic formula simplifies the evaluation of the position of the maximum value of a Zolotarev polynomial (21). Its matrix form (94) was successfully used for an efficient code in Matlab. The evaluation of the discrete zeta function uses the standard procedure for the elliptic function sn.

```
function u=zeta(n,k)
% ******************************************
% * zeta(n,k)
% * Jacobi's Zeta Function of discrete
% * argument K(k)/n
% * evaluation based on addition theorem
% * Z(u) + Z(v) - Z(u+v) =
% *        k*k*sn(u|k)*sn(v|k)*sn(u+v|k)
% * Erlangen, June 1997, Miroslav Vlcek
% ******************************************
quarter=ellipke(k.*k);
s=ellipj((1:n)*quarter/n, k.*k);
v=s(n-(1:n-1)).*s(n+1-(1:n-1));
a=diag(n-1:-1:1)*ones(n-1);
b=ones(n-1)-tril(ones(n-1));
u=k.*k*s(1)/n*(a-n*b)*v';
```

The elliptic integral of the third kind $\Pi(u, a\kappa)$ present a far more formidable computational problem on account of its dependence on three parameters. In our application the parameter a is attributed to the specific discrete values of the half-period for which the addition formula holds

$$\Pi(u,p|\kappa) + \Pi(u,r|\kappa) - \Pi(u,p+r|\kappa) =$$
$$\frac{1}{2}\ln\frac{1-\kappa^2 sn(\frac{p}{n}\,\mathbf{K})sn(\frac{r}{n}\,\mathbf{K})sn(u)sn(\frac{p+r}{n}\,\mathbf{K}-u)}{1+\kappa^2 sn(\frac{p}{n}\,\mathbf{K})sn(\frac{r}{n}\,\mathbf{K})sn(u)sn(\frac{p+r}{n}\,\mathbf{K}+u)} +$$
$$+u\,\kappa^2 sn(\frac{p}{n}\,\mathbf{K})sn(\frac{r}{n}\,\mathbf{K})sn(\frac{p+r}{n}\,\mathbf{K}) \equiv R(u,p,r|\kappa)$$

The addition formula for parameters has a similar form to that of the zeta function so we can immediately write the algebraic formula [10]

$$[\Pi] = \frac{1}{n}\left(\mathbf{A} - n\mathbf{B}\right)[R], \tag{100}$$

where the abbreviated notation for vectors is introduced

$$[\Pi] = \begin{bmatrix} \Pi(u, \frac{n-1}{n}\,\mathbf{K}|\kappa) \\ \vdots \\ \Pi(u, \frac{2}{n}\,\mathbf{K}|\kappa) \\ \Pi(u, \frac{1}{n}\,\mathbf{K}|\kappa) \end{bmatrix} \tag{101}$$

$$[R] = \begin{bmatrix} R(u,n-1,1|\kappa) \\ \vdots \\ R(u,2,1|\kappa) \\ R(u,1,1|\kappa) \end{bmatrix}. \tag{102}$$

The notation in equation (100) is the same as in equation (94). The algebraic representation of the elliptic integral of the third kind of the discrete parameter (100) reduces the evaluation of the maximum value of a Zolotarev polynomial to the standard elliptic function sn. The formula was used for an efficient code in Matlab.

```
function f=ellipi(u,n,k)
% ******************************************
% * f=ellipi(u,n,k)                        *
% * Elliptic integral of the third kind    *
% * of discrete parameter K(k)/n,          *
% * argument u and modulus k               *
% * evaluation based on addition theorem   *
% * for parameters                         *
% * P(u,a) + P(u,b) - P(u,a+b) = R(u,a,b)  *
% * Erlangen, July 1997, Miroslav Vlcek    *
% ******************************************
quarter=ellipke(k*k);
si=ellipj(u,k.*k);
s=ellipj((1:n)*quarter/n,k*k);
sp=ellipj((1:n)*quarter/n+u,k*k);
sm=ellipj((1:n)*quarter/n-u,k*k);
v=u*k*k*s(1)*s(n-(1:n-1)).*s(n+1-(1:n-1));
nu=1-k*k*s(1)*si*s(n-(1:n-1)).*sm(n+1-(1:n-1));
de=1+k*k*s(1)*si*s(n-(1:n-1)).*sp(n+1-(1:n-1));
r=log(nu./de)/2 + v;
a=diag(n-1:-1:1)*ones(n-1);
b=ones(n-1)-tril(ones(n-1));
f=flipud(1/n*(a-n*b)*r');
```

X. Appendix II - Relation between Chebyshev and Zolotarev polynomials

The Chebyshev polynomials of the first kind $T_n(x)$ are defined as

$$T_n(x) = \frac{1}{2}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right]. \tag{103}$$

As the following relation holds

$$(x+\sqrt{x^2-1})(x-\sqrt{x^2-1}) = 1, \tag{104}$$

we can rewrite equation (103)

$$T_n(x) = \frac{1}{2}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right] = \frac{1}{2}(\lambda^n + \lambda^{-n})\,. \quad (105)$$

It is clear that

$$x = \frac{1}{2}(\lambda + \lambda^{-1})\,, \quad (106)$$

and finally we obtain the formula

$$T_n\left(\frac{1}{2}(\lambda+\lambda^{-1})\right) = \frac{1}{2}(\lambda^n + \lambda^{-n}) \quad (107)$$

which gives a straightforward relation of a Zolotarev polynomial to the Chebyshev polynomial, eqs. (25) and (26).

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Miroslav Vlček was born in Prague, Czech Republic, in 1951. He graduated in theoretical physics from Charles University, Prague, in 1974. In 1979, he received the Ph.D. degree in communication engineering and in 1994 the D.Sc. degree from Czech Technical University in Prague (CTU). He has been with the Department of Circuit Theory of the Faculty of Electrical Engineering of CTU from 1974 to 1993. Since 1994 he is the head of the Department of Applied Mathematics of the Faculty of Transportation Sciences. Recently he teaches courses in system theory and digital filter design. His scientific interests include filter design and digital signal processing, theory of approximation and higher transcendental functions.

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Rolf Unbehauen (M'61-SM'82) was born in Stuttgart, Germany in 1930. He received the Diplom-Mathematiker and the Dr.-Ing. degrees and the venia legendi in electrical engineering from Stuttgart University, Stuttgart, Germany, in 1954, 1957 and 1964, respectively. In 1959 he received the NTG Best Paper Award, in 1994 he obtained an honorary doctorate from Technical University Cluj-Napoca, Romania. He was employed with the Mathematical Institute, the Computation Center, and the Institute of Electrical Engineering at Stuttgart University from 1955 to 1964, and then appointed Wissenschaftlicher Rat. Since 1966 he has been Professor of Electrical Engineering at the University of Erlangen-Nürnberg in Erlangen, Federal Republic of Germany. He is currently teaching courses and doing research work in system theory, network theory, and field theory. Dr. Unbehauen is a member of the Informationstechnische Gesellschaft of Germany and of URSI, Comission C: Signals and Systems. Currently he is an Associate Editor of the *IEEE Transactions on Neural Networks* and *Multidimensional System and Signal Processing.*