CHAPTER 2

SEQUENCES

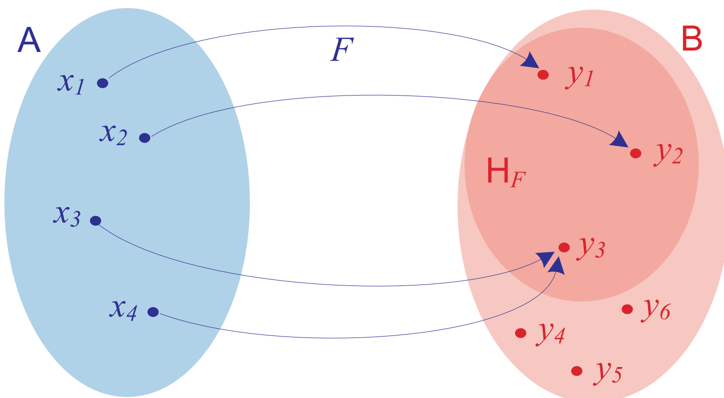
Mapping

Definition 1. Consider two non-empty sets A, B. **A mapping of a set A to B** is defined as a set F of ordered pairs $(x, y) \in$ **A** $\times$ **B**, where for every $x \in$ **A** there exists **exactly one** element $y \in$ **B** such that $(x, y) \in F$.

An element x is called a **preimage** of an element y, an element y is called an **image** of x in the mapping F. We also say that y is the **value** of the mapping F in a point x and write $y = F(x)$ or $x \mapsto F(x)$. A set **A** is called a **domain of a mapping** F and it is also denoted by a symbol $\mathbf{D}(F)$ or $\mathbf{D}_F$. The set of all images in the mapping F is called **range of the mapping** F and it is denoted by $\mathbf{H}(F)$ or $\mathbf{H}_F$. It is $\mathbf{H}(F) \subset \mathbf{B}$.

Symbolically, a mapping F from **A** to **B** is expressed as follows:

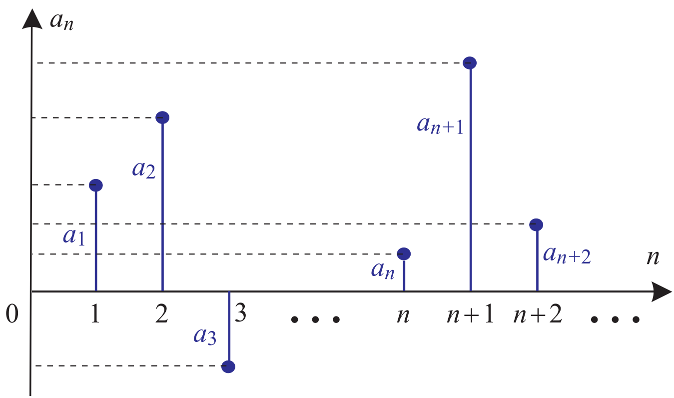
$$F: \mathbf{A} \to \mathbf{B}, \quad \mathbf{D}(F) = \mathbf{A}$$



Sequence of real numbers

Definition 2. A sequence (a_n) of real numbers $f : \mathbb{N} \to \mathbb{R}$, where $a_n = f(n)$.

A sequence therefore assigns a unique element $a_n = f(n) \in \mathbb{R}$, called **term of a sequence**, to every $n \in \mathbb{N}$. The whole sequence is usually denoted by (a_n). A graph of a sequence consists of isolated points:



☛ ***Example 1.***

Arithmetic sequence is defined by a formula:

$$a_1 \in \mathbb{R}\,, \qquad a_n = a_1 + (n-1)d\,,$$

where $a_1\,, d$ are given real numbers.

Terms of an arithmetic sequence satisfy the condition: $a_{n+1} - a_n = d\,.$

A number d is called **difference of an arithmetic sequence.**

By mathematical induction it can be proved that the sum of the first n terms of an arithmetic sequence satisfy the equation:

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n = \frac{n\big(a_1 + a_n\big)}{2} = na_1 + \frac{n(n-1)}{2}\,d\,.$$

We can also consider:

$$\begin{array}{rcllll} s_n & = & a_1 & +(a_1+d) & +(a_1+2d) + \cdots + & (a_1+(n-1)d) \\ s_n & = & a_n & +(a_n-d) & +(a_n-2d) + \cdots + & (a_n-(n-1)d) \\ \hline 2s_n & = & (a_1+a_n) & +(a_1+a_n) & +(a_1+a_n) + \cdots + & (a_1+a_n) \end{array}$$

$$\Longrightarrow 2s_n = n(a_1+a_n) \Longrightarrow s_n = \tfrac{1}{2}n(a_1+a_n)$$

☛ ***Example 2.***

Geometric sequence is defined by a formula

$$a_1 \in \mathbb{R}\,, \qquad a_n = a_1 q^{n-1}\,,$$

where $a_1\,, q$ are given real numbers.

If $a_1 q \neq 0$, the equation

$$\frac{a_{n+1}}{a_n} = q\,.$$

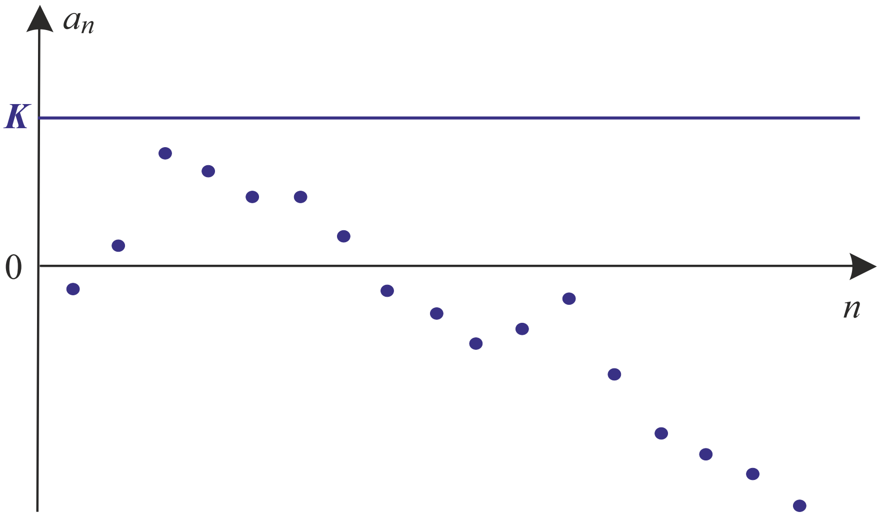
holds for all $n \in \mathbb{N}$. This ratio is called **quotient of a geometric sequence.**

By mathematical induction it can be proved that the sum of the first n elements of a geometric sequence satisfy the equation:

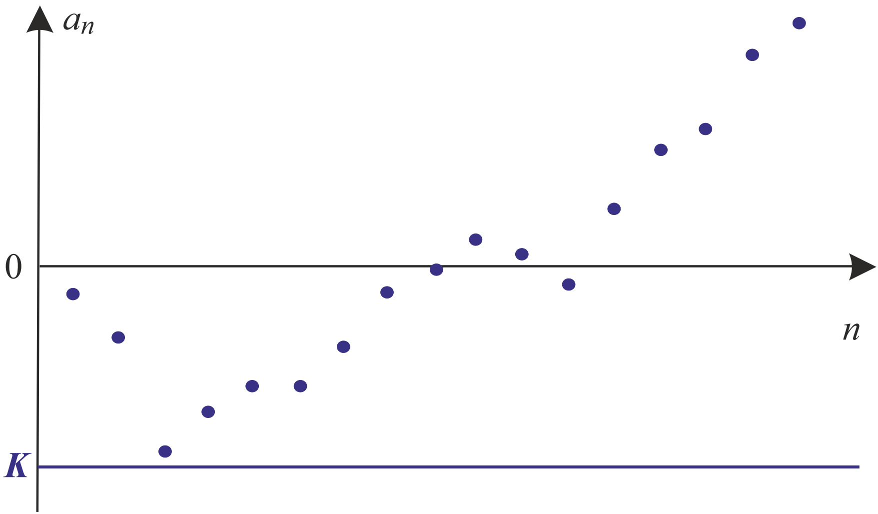
$$s_n = \sum_{k=1}^{n} a_k = a_1\left(1 + q + q^2 + \cdots + q^{n-1}\right) = \begin{cases} a_1 \dfrac{q^n - 1}{q - 1} & \text{pro } q \neq 1, \\ n a_1 & \text{for } q = 1. \end{cases}$$

Properties of Sequences

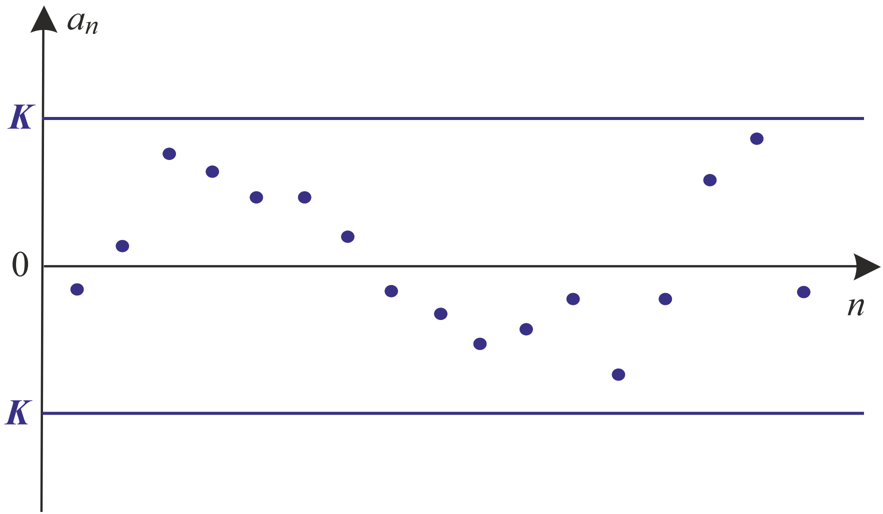
Definition 3. A sequence (a_n) is called **bounded from above**, it there exists $K \in \mathbb{R}$ such that $a_n \leq K$ for all $n \in \mathbb{N}$.



Definition 4. A sequence (a_n) is called **bounded from below**, it there exists $K \in \mathbb{R}$ such that $a_n \geq K$ for all $n \in \mathbb{N}$.



Definition 5. A sequence (a_n) is called **bounded from above**, it there exists $K \in \mathbb{R}$ such that $|a_n| \leq K$ for all $n \in \mathbb{N}$.



☛ ***Example 3.***

Let $d > 0$. An arithmetic sequence (a_n) is bounded from bellow by a_1, but it is not bounded from above, and thus it is not bounded.

☛ ***Example 4.***

Consider a geometric sequence (a_n) with $a_1 \neq 0$.

If $q < -1$ then it is bounded neither from bellow nor above.

If $|q| = 1$ then it is bounded (e.g., consider $K = |a_1|$).

If $q > 1$ then it is bounded from below (e.g., $K = |a_1|$).

Definition 6. A sequence (a_n) is called

- ➥ **increasing** if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$,
- ➥ **decreasing** if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$,
- ➥ **non-decreasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$,
- ➥ **non-increasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

A sequence satisfying one of the above-stated conditions is called **monotone.** If it is increasing or decreasing, it is also called **strictly monotone.**

☛ ***Example 5.***

Consider a sequence (a_n), where $a_n = \dfrac{(-1)^{n+1}}{n}$.

Terms of the sequence: $1\,,\ -\frac{1}{2}\,,\ \frac{1}{3}\,,\ -\frac{1}{4}\,,\ \frac{1}{5}\,,\ -\frac{1}{6}\,,\ \ldots\,.$

This sequence is not monotone, it is bounded by 1.

Definition 7. Consider a sequence (a_n) and an increasing sequence of natural numbers (k_n), i.e.,

$$k_n \in \mathbb{N} \qquad \text{a} \qquad k_n < k_{n+1} \, .$$

A sequence (b_n), where $b_n = a_{k_n}$, is called a **subsequence** of a sequence (a_n).

☛ ***Example 6.***

A sequence (b_n) defined by the equation

$$b_n = \frac{(-1)^{n^2+1}}{n^2}$$

is a subsequence of a sequence (a_n), where

$$a_n = \frac{(-1)^{n+1}}{n} \, .$$

In this case, $k_n = n^2$ $(b_1 = 1 = a_1; \; b_2 = -\frac{1}{4} = a_4 \; \ldots)$.

☛ ***Example 7.***

A sequence (c_n), where $c_n = \dfrac{1}{n^2}$, is not a subsequence of (a_n), where

$$a_n = \frac{(-1)^{n+1}}{n},$$

since no increasing sequence of natural numbers (k_n) exists such that $a_{k_n} = c_n = \frac{1}{n^2} \quad (c_1 = 1 = a_1;\ c_2 = \frac{1}{4},\ a_4 = -\frac{1}{4})$.

☛ ***Example 8.***

A sequence (d_n) with terms

$$1\,,\ -\frac{1}{2}\,,\ \frac{1}{3}\,,\ \frac{1}{5}\,,\ -\frac{1}{4}\,,\ \frac{1}{7}\,,\ \frac{1}{9}\,,\ -\frac{1}{6}\,,\ \frac{1}{11}\,,\ \dots$$

is not selected from a sequence (a_n), even though the sets of terms of both sequences are equal.

Algebraic operations

Multiplication of a sequence (a_n) by a real number $c \in \mathbb{R}$:

$$c(a_n) = (ca_n) \,.$$

A sum of sequences (a_n) and (b_n) :

$$(a_n) + (b_n) = (a_n + b_n) \,.$$

A difference of sequences (a_n) and (b_n) :

$$(a_n) \cdot (b_n) = (a_n \cdot b_n) \,.$$

A quotient of sequences (a_n), (b_n), where $b_n \neq 0$ for all $n \in \mathbb{N}$:

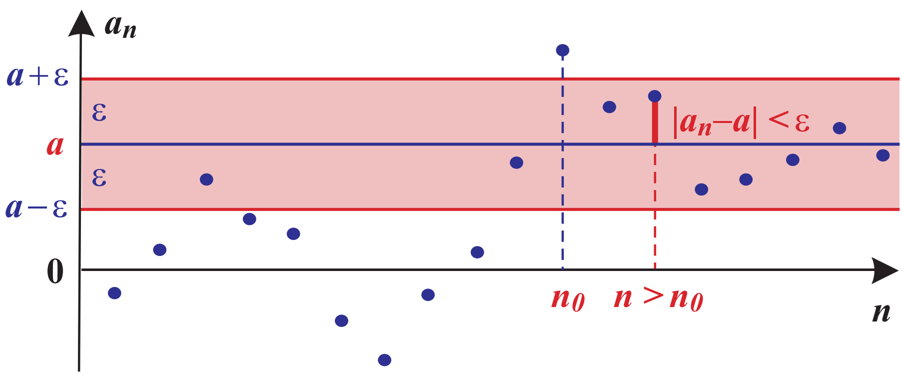
$$\frac{(a_n)}{(b_n)} = \left(\frac{a_n}{b_n} \right) \,.$$

Limit of a sequence

Definition 8. We say that a sequence (a_n) has a **limit** $a \in \mathbb{R}^*$, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $a_n \in U_\varepsilon(a)$ for all $n > n_0$.

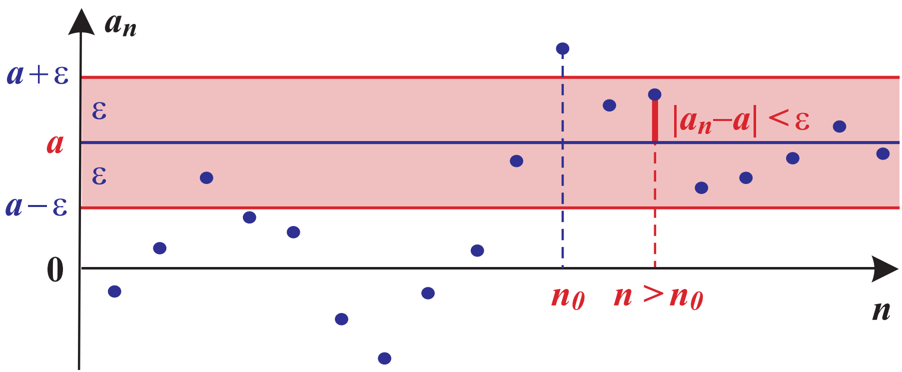
We write $\lim_{n\to\infty} a_n = a$ or simply $\lim a_n = a$.

Notice that for $a \in \mathbb{R}$, $a_n \in U_\varepsilon(a)$ means that $|a_n - a| < \varepsilon$. In this case, we speak about a **proper limit.**



A proper limit can also be defined separately as follows.

Definition 9. We say that a sequence (a_n) has a **proper limit** $a \in \mathbb{R}$, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n > n_0$.



☛ ***Example 9.***

Prove that $\lim\limits_{n\to\infty} \dfrac{n+4}{n^3+n+1} = 0$.

Solution. Let $\varepsilon > 0$ is given. An inequality

$$\frac{n+4}{n^3+n+1} < \frac{5n^2}{n^3} = \frac{5}{n}$$

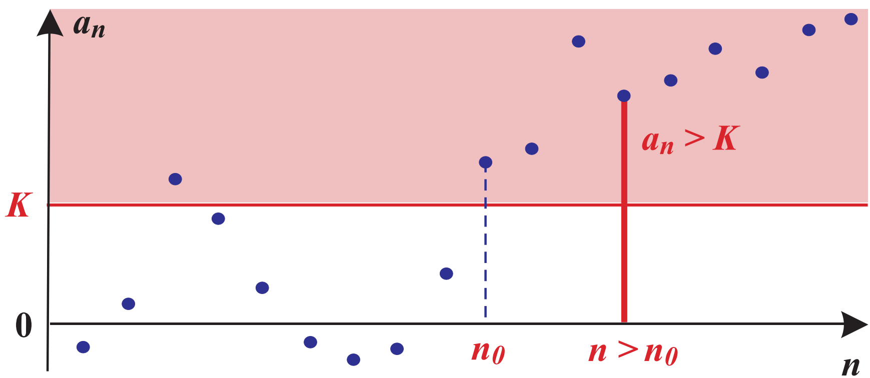
implies that for $n_0 \in \mathbb{N}$ such that $\dfrac{5}{n_0} < \varepsilon$, the following equation holds for all $n \in \mathbb{N}$, $n > n_0$:

$$\left|\frac{n+4}{n^3+n+1} - 0\right| = \frac{n+4}{n^3+n+1} < \frac{5}{n} < \frac{5}{n_0} < \varepsilon\,.$$

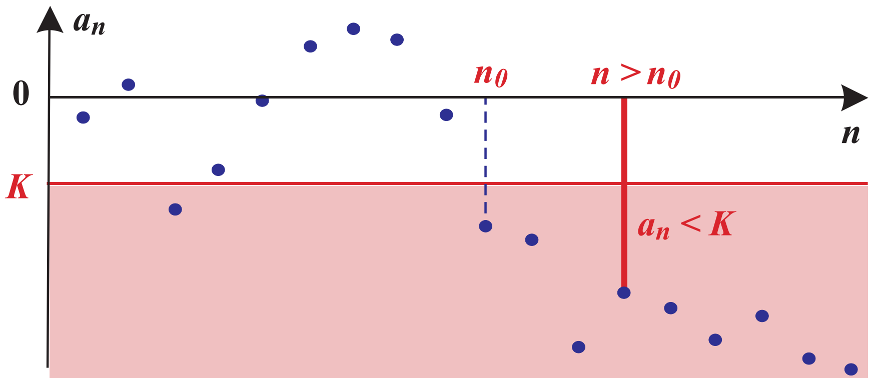
It is sufficient to consider $n_0 = \left[\dfrac{5}{\varepsilon}\right] + 1$, where $[x]$ is a so-called whole part of a real number x, which is defined for any $x \in \mathbb{R}$ as a unique integer satisfying the inequalities $[x] \le x < [x] + 1$.

If a limit a is infinite, it is called **improper.**
In this case, $a_n \in U_\varepsilon(+\infty)$ means that $a_n > 1/\varepsilon$ and $a_n \in U_\varepsilon(-\infty)$ means that $a_n < -1/\varepsilon$. It can also be defined separately:

Definition 10. We say that a sequence (a_n) has an **improper limit** $+\infty$, if for every $K \in \mathbb{R}$ there exists n_0 such that $a_n > K$ for all $n > n_0$.



Definition 11. We say that a sequence (a_n) has an **improper limit** $-\infty$, if for every $K \in \mathbb{R}$ there exists n_0 such that $a_n < K$ for all $n > n_0$.



Theorem 1. *Every sequence has at most one limit.*

Proof. By contradiction:

If (a_n) had two different limits a and b, $a \neq b$, it would be possible to choose disjoint neighbourhoods of these points, $U_{\varepsilon_1}(a)$ and $U_{\varepsilon_2}(b)$ with $U_{\varepsilon_1}(a) \cap U_{\varepsilon_2}(b) = \emptyset$.

For finite a, $b \in \mathbb{R}$ we can consider e.g. $\varepsilon_1 = \varepsilon_2 = \dfrac{|a-b|}{3} > 0$.

From definition of a limit: there exists $n_1 \in \mathbb{N}$ such that $a_n \in U_{\varepsilon_1}(a)$ for all $n > n_1$, and n_2 such that $a_n \in U_{\varepsilon_2}(a)$ for all $n > n_2$.

But then $a_n \in U_{\varepsilon_1}(a) \cap U_{\varepsilon_2}(b) = \emptyset$ for all $n > \max\bigl(n_1, n_2\bigr)$.

The assumption that a is different from b therefore leads to a contradiction, thus it must be $a = b$. □

Definition 12. If a sequence (a_n) has a proper limit, it is called **convergent**. Otherwise (i.e., its limit is improper or does not exist) it is called **divergent.**

Theorem 2. *Every convergent sequence is bounded.*

Proof. Let $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$. Let us choose $\varepsilon = 1$. Then there exists $n_0 \in \mathbb{N}$ such that $a - 1 < a_n < a + 1$ for all $n > n_0$. Denote

$$K = \max\{a_1, a_2, \ldots, a_{n_0}, a+1\}, \ L = \min\{a_1, a_2, \ldots, a_{n_0}, a-1\}.$$

These values K and L exist, since they represent maximum and minimum of a finite set, respectively, and the inequality $L \leq a_n \leq K$ holds for all $n \in \mathbb{N}$. $\square$

Theorem 3. *Let* (a_n) *and* (b_n) *be convergent sequences,* $c \in \mathbb{R}$. *Let* $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$.

Then the sequences

$$(ca_n)\,,\ (a_n + b_n)\,,\ (a_n \cdot b_n)$$

converge, too, and the following equations hold:

$$\lim_{n\to\infty} (ca_n) = ca\,, \quad (a_n + b_n) = a + b\,, \quad \lim_{n\to\infty} (a_n \cdot b_n) = ab\,.$$

If $\lim\limits_{n\to\infty} b_n \neq 0$, *then the sequence* $\left(\dfrac{a_n}{b_n}\right)$ *convergs to the limit*

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}\,.$$

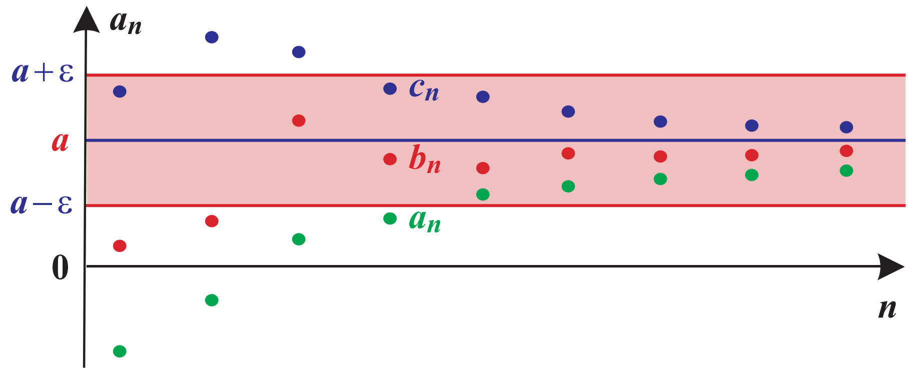
Theorem 4. *Let (a_n) and (b_n) be convergent sequences such that $a_n \le b_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.*

Proof. Denote $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$.

If $a > b$, then for $\varepsilon = \frac{a-b}{2} > 0$ there exist $n_a,\ n_b$ such that $a - \varepsilon = \frac{a+b}{2} < a_n$ for all $n > n_a$ and $b_n < b + \varepsilon = \frac{a+b}{2}$ for all $n > n_b$. Thus $b_n < \frac{a+b}{2} < a_n$ for all $n > \max(n_a, n_b)$, which is a contradiction. □

Remark: The limits a and b can be equal, $a = b$, even if $a_n < b_n$ for all $n \in \mathbb{N}$. For example: $a_n = 0,\ b_n = \frac{1}{n}$.

Theorem 5. *Let sequences (a_n), (b_n), (c_n) be such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If limits of (a_n) and (c_n) exist and are equal, i.e., $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = a$, then the limit of a sequence (b_n) exists, too, and is equal to $\lim\limits_{n\to\infty} b_n = a$.*



Proof. The assumption is obvious for $\lim\limits_{n\to\infty} a_n = +\infty$ or $\lim\limits_{n\to\infty} c_n = -\infty$. Let $a \in \mathbb{R}$. Then for each $\varepsilon > 0$ there exist n_a, n_c such that $a - \varepsilon < a_n$ for all $n > n_a$ and $c_n < a + \varepsilon$ for all $n > n_b$. Thus $a - \varepsilon < a_n \leq b_n \leq c_n < a + \varepsilon$ for all $n > n_0 = \max(n_a, n_b)$. $\square$

Theorem 6. *For any sequence* (a_n), $\lim_{n\to\infty} a_n = 0$ *if and only if* $\lim_{n\to\infty} |a_n| = 0$.

Proof. The proposition follows directly from the definition of a limit. □

Theorem 7. *If* $\lim_{n\to\infty} a_n = 0$ *and a sequence* (b_n) *is bounded, then* $\lim_{n\to\infty} a_n b_n = 0$.

Proof. Since $\lim_{n\to\infty} a_n = 0$, it is also $\lim_{n\to\infty} |a_n| = 0$. Since a sequence (b_n) is bounded, there exists $K \in \mathbb{R}$ such that $-K \le b_n \le K$ for all $n \in \mathbb{N}$. Obviously, $-K|a_n| \le |a_n b_n| \le K|a_n|$, $\lim_{n\to\infty} K|a_n| = 0$.
□

☛ ***Example 10.***

Find the limit of a sequence $a_n = \dfrac{\sin n!}{n}$.

Solution: Denote

$$a_n = b_n \cdot c_n, \quad \text{where} \quad b_n = \frac{1}{n}, \; c_n = \sin n!.$$

Obviously,

$\lim b_n = 0;$

$\left|\sin n!\right| \leq 1.$

A sequence (c_n) is therefore bounded and the previous theorem imply that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n c_n = 0$, i.e.,

$$\lim_{n\to\infty} \frac{\sin n!}{n} = 0.$$

☛ ***Example 11.***

Find the limit of a sequence $a_n = \dfrac{2^{\cos n}}{n + \sin n!}$.

Solution: Denote

$$a_n = b_n \cdot c_n, \quad \text{where} \quad b_n = \frac{1}{n}, \; c_n = \frac{2^{\cos n}}{1 + \left(\sin n!\right)/n}.$$

Obviously,

$\lim b_n = 0;$

$|\cos n| \le 1 \Rightarrow 2^{\cos n} \le 2;$

$\left|\sin n!\right| \le 1 \Rightarrow \lim\limits_{n\to\infty} \dfrac{\sin n!}{n} = 0 \Rightarrow \lim\limits_{n\to\infty} \left(1 + \dfrac{\sin n!}{n}\right) = 1.$

A sequence $\left(c_n\right)$ is therefore bounded and the previous theorem imply that $\lim\limits_{n\to\infty} a_n = 0$.

Theorem 8. *If a sequence (a_n) is non-decreasing, then its limit (either proper or improper) exits and is equal to*

$$\lim_{n\to\infty} a_n = \sup a_n \,.$$

If a sequence (a_n) is non-increasing, then its limit (either proper or improper) exits and is equal to

$$\lim_{n\to\infty} a_n = \inf a_n \,.$$

Remark: In other words, the theorem says that a monotone sequence always has a limit (proper or improper), and that this limit is equal to its supremum (for a non-decreasing sequence) or infimum (for a non-increasing sequence).

Proof. Suppose first that a sequence (a_n) **is non-decreasing,** i.e., $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

If (a_n) **is not bounded,** than for any $K \in \mathbb{R}$ there exists n_0 such that $a_{n_0} > K$. Since the sequence is non-decreasing, the inequality $K < a_{n_0} \leq a_n$ holds for all $n > n_0$. Thus

$$\lim_{n\to\infty} a_n = +\infty.$$

If (a_n) **is bounded from above** (notice that it is always bounded from below), then there exists a finite $\sup\{a_n\,;\ n \in \mathbb{N}\} = a \in \mathbb{R}$. We show that it is also a limit of the sequence (a_n).

A supremum is an **upper-bound,** thus $a_n \leq a$ for all $n \in \mathbb{N}$. Consider any $\varepsilon > 0$. Since a supremum is the **least** upper bound, there exists $n_0 \in \mathbb{N}$ such that $a - \varepsilon < a_{n_0} \leq a$. Since (a_n) is non-decreasing, the inequality $a - \varepsilon < a_{n_0} \leq a_n \leq a$ holds for all $n > n_0$. Thus

$$\lim_{n\to\infty} a_n = a.$$

For a non-increasing sequence, the proof is analogous. □

On the basis of this theorem, the following important relations can be proved:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \mathrm{e}, \quad \text{more general,} \quad \lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = \mathrm{e}^k$$

☛ ***Example 12.*** Prove that $a_n = \left(1+\frac{1}{n}\right)^n$ is convergent.

Denote $b_n = (1+1/n)^{n+1}$. This sequence is decreasing:

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^{n+1} &> \left(1+\frac{1}{n+1}\right)^{n+2} \\
\left(\frac{n+1}{n}\right)^{n+1} &> \left(\frac{n+2}{n+1}\right)^{n+2} && \Big| \cdot \frac{n+1}{n} \\
\left(\frac{n+1}{n}\right)^{n+2} &> \left(\frac{n+2}{n+1}\right)^{n+2} \cdot \frac{n+1}{n} && \Big| : \left(\frac{n+2}{n+1}\right)^{n+2} \\
\left(\frac{(n+1)^2}{n(n+2)}\right)^{n+2} &> 1+\frac{1}{n}
\end{aligned}$$

The last inequality is true, since

$$\left(\frac{(n+1)^2}{n(n+2)}\right)^{n+2} = \left(\frac{n^2+2n+1}{n^2+2n}\right)^{n+2} = \left(1+\frac{1}{n(n+2)}\right)^{n+2} =$$

$$= 1+1\cdot(n+2)\cdot\frac{1}{n(n+2)}+1\cdot\binom{n+2}{2}\cdot\left(\frac{1}{n(n+2)}\right)^2+\cdots > 1+\frac{1}{n}\,.$$

A sequence (b_n) is therefore decreasing. Since $b_n > 0$ for all n, this sequence is bounded from bellow and has a proper limit. Let us denote this limit by e. Since

$$\left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right),$$

the sequence (a_n) has the same limit, called **Euler's number:**

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e = 2{,}718\ 281\ 828\ 459\ 045\ \ \ldots\ .$$

☛ ***Example 13.*** Prove that

$$\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{kn} = e^k\,.$$

Similarly as in the previous example, it can be shown that for any $k \in \mathbb{N}$, a sequence $a_n = (1+k/n)^{n+k}$ is decreasing and bounded from bellow, thus its limit exists.

We can select a subsequence $(b_m) = (a_{km}) = (1+1/m)^{km+k}$. According to the previous example,

$$\lim_{m\to\infty}(1+1/m)^{km+k} = \left(\lim_{m\to\infty}(1+1/m)^m\right)^k \cdot \lim_{m\to\infty}(1+1/m)^k = e^k.$$

Since the limit of (a_n) exists, it is

$$\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k.$$

☛ ***Example 14.*** Find the limit $\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{6n+5}$.

Solution. Denote $a_n = \left(1+\frac{1}{3n}\right)^{6n+5}$. Consider a sequence $b_m = (1+1/m)^{2m+5}$. Obviously $b_{3n} = a_n$, thus (a_n) is a subsequence of (b_n). It is

$$\left(1+\frac{1}{m}\right)^{2m+5} = \left(1+\frac{1}{m}\right)^{2m} \cdot \left(1+\frac{1}{m}\right)^{5}.$$

Since the limit of the first factor is equal to e^2 and the limit of the second factor is equal to 1,

$$\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{6n+5} = e^2\,.$$

Later we will prove: If $\lim\limits_{n\to\infty} a_n = 0$, $\lim\limits_{n\to\infty} b_n = +\infty$, then

$$\lim_{n\to\infty}(1+a_n)^{b_n} = e^{\alpha},\ \text{where } \alpha = \lim_{n\to\infty} a_n b_n.$$

Definition 13. A sequence (a_n) is called a **Cauchy sequence**, if it satisfies the **Bolzano–Cauchy condition**: For any $\varepsilon > 0$ there exist n_0 such that $|a_m - a_n| < \varepsilon$ for all m, n, where $m > n_0$ and $n > n_0$.

Theorem 9. *A sequence (a_n) is convergent if and only if it is a Cauchy sequence.*

Theorem 10. *Let (b_n) be a subsequence of a sequence (a_n) with $\lim\limits_{n\to\infty} a_n = a$. Then $\lim\limits_{n\to\infty} b_n = a$.*

Proof. For any $\varepsilon > 0$ (or $K \in \mathbb{R}$), it is sufficient to choose $n_0 = k_{n_0}$.

☛ ***Example 15.***

Prove that a sequence with $a_n = (-1)^n$ does not have a limit.

Solution. For $n = 2k$ we get a subsequence $b_k = a_{2k} = (-1)^{2k} = 1$ with a limit equal to 1, for $n = 2k + 1$ we get a subsequence $b_k = a_{2k+1} = (-1)^{2k+1} = -1$ with a limit -1.

☛ ***Example 16.***

Prove that a sequence $a_n = \left(1 + \frac{(-1)^n}{n}\right)^n$ does not have a limit.

Solution. For even $n = 2k$, we get a subsequence

$$b_k = a_{2k} = \left(1 + \frac{1}{2k}\right)^{2k}.$$

It is a subsequence of a sequence $\left(1 + \frac{1}{n}\right)^n$, therefore $\lim\limits_{k\to\infty} b_k = e$.
For odd $n = 2k-1$, we get a subsequence

$$c_k = a_{2k-1} = \left(1 - \frac{1}{2k-1}\right)^{2k-1},$$

which is a subsequence of a sequence $\left(1 - \frac{1}{n}\right)^n$. Since all terms of this sequence are less than 1, its limit cannot be equal to $e > 1$. Actually, $\lim\limits_{k\to\infty} c_k = e^{-1}$. Since the sequence (a_n) contains two subsequences with different limits, its limit does not exist.

Definition 14. A point $a \in \mathbb{R}^*$ is called **an accumulation point** of a sequence (a_n) if and only if there exists a subsequence (b_n) of a sequence (a_n) such that $a = \lim\limits_{n\to\infty} b_n$.

Theorem 11. *A point a is an accumulation point of a sequence (a_n) if and only if for each $U_\varepsilon(a)$ there exists an infinite set $N_a \subset \mathbb{N}$ such that $a_n \in U_\varepsilon(a)$ for all $n \in N_a$.*

Proof. The theorem is just a rephrased definition of an accumulation point of a sequence. □

☛ ***Example 17.***

For a sequence $a_n = (-1)^n$, accumulation points are 1 and -1, since

$$\lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} 1 = 1, \quad \lim_{k\to\infty} a_{2k-1} = \lim_{k\to\infty} (-1) = -1.$$

☛ ***Example 18.***

Find all accumulation points of a sequence

$$a_n = \frac{(n+1)^2 + (-1)^n n^2}{n^2+n+1} \cdot \cos\left(\tfrac{2}{3}\pi n\right) .$$

Solution. $a_n = b_n \cdot c_n$, where

$$b_n = \frac{(n+1)^2 + (-1)^n n^2}{n^2+n+1}, \qquad c_n = \cos\left(\tfrac{2}{3}\pi n\right) .$$

Neither of these sequences has a limit.

$$b_{2k} = \frac{8k^2+4k+1}{4k^2+6k+1} \to 2, \qquad b_{2k-1} = \frac{4k-1}{4k^2-2k+1} \to 0 .$$

Since a sequence c_n is bounded, it is $\lim\limits_{k\to\infty} a_{2k-1} = 0$.

Consider

$$a_{2k} = \frac{8k^2+4k+1}{4k^2+6k+1} \cdot \cos\left(\tfrac{2}{3}\pi k\right);$$

$\cos\left(\frac{4}{3}\pi k\right)$ is equal to 1 for $k = 3m$ and $-\frac{1}{2}$ for $k = 3m \pm 1$. A sequence (a_{2k}) has therefore a subsequence $(a_{6k})$with a limit 2 and a subsequence $(a_{6k\pm 2})$ with a limit -1. Accumulation points of (a_n) are therefore -1, 0 and 2.

☛ ***Example 19.***

Find all accumulation points of a sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-2}{n}, \frac{n-1}{n}, \ldots.$$

Solution. This sequence contains all rational numbers from the interval $(0,1)$, i.e., all fractions $\frac{p}{q}$, where $0 < p < q$ are natural, mutually prime numbers. Since any real number can be approximated by a sequence of rational numbers (with an arbitrary accuracy), the set of accumulation points of a sequence (a_n) is the whole interval $\langle 0, 1\rangle$.

Definition 15. Let M be a set of all accumulation points of a sequence (a_n). The number $S = \sup M$ is called **limes superior** of a sequence (a_n) and it is denoted by $\limsup\limits_{n\to\infty} a_n$ or $\overline{\lim\limits_{n\to\infty}}\, a_n$. The number $s = \inf M$ is called **limes inferior** of a sequence (a_n) and it is denoted by $\liminf\limits_{n\to\infty} a_n$ or $\underline{\lim\limits_{n\to\infty}}\, a_n$.

☛ ***Example 20.***

For a sequence $a_n = (-1)^n$, limes superior and limes inferior are

$$\overline{\lim_{n\to\infty}}\,(-1)^n = 1, \qquad \underline{\lim_{n\to\infty}}\,(-1)^n = -1.$$

Theorem 12. *A sequence (a_n) has a limit if and only if*

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n\,.$$

Theorem 13. *A set M is compact if and only if from each sequence (a_n), where $a_n \in M$ for all $n \in \mathbb{N}$, a subsequence can be selected such that its limit lies in M.*