CHAPTER 8

# REAL FUNCTIONS OF MORE VARIABLES

# Limit of a function of $n$ real variables

**Definition 1.** Let $\boldsymbol{f} : D_{\boldsymbol{f}} \subset \mathbb{R}^n \to \mathbb{R}^k$ be a mapping, let $\boldsymbol{a}$ be an accumulation point of its domain $D_{\boldsymbol{f}}$. We say that $\boldsymbol{A} \in \mathbb{R}$ is **a limit** of the mapping $\boldsymbol{f}$ at the point $\boldsymbol{a}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\boldsymbol{f}(\boldsymbol{x}) \in U_\varepsilon(\boldsymbol{A})$ for all $\boldsymbol{x} \in P_\varepsilon(\boldsymbol{a})$.

**Definition 2.** Let $\boldsymbol{f} : D_{\boldsymbol{f}} \subset \mathbb{R}^n \to \mathbb{R}^k$ be a mapping, **M** $\subset D_{\boldsymbol{f}}$. We say that the function $\boldsymbol{f}(\boldsymbol{x})$ has **a limit A at the point $\boldsymbol{a}$ with respect to the set M** if the function $\widehat{\boldsymbol{f}} = \boldsymbol{f}\big|_M$ has a limit $\boldsymbol{A}$ at $\boldsymbol{a}$. We write

$$\lim_{\substack{x \to \boldsymbol{a} \\ x \in M}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\,.$$

Any mapping $\boldsymbol{f} : D_f \subset \mathbb{R}^n \to \mathbb{R}^k$ is given by the formula

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}\big(x_1, x_2, \dots, x_n\big) = \big(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \dots, f_k(\boldsymbol{x})\big)\,,$$

where $f_i : X \to \mathbb{R}$, $i = 1, 2, \dots, k$, are real functions of $n$ variables.

**Theorem 1.** *Let $\boldsymbol{f} : X \to \mathbb{R}^k$, where $\boldsymbol{X} \subset \mathbb{R}^n$. Then*

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}$$

*if and only if $\lim\limits_{\boldsymbol{x}\to\boldsymbol{a}} f_i(\boldsymbol{x}) = A_i$ for all $i = 1, 2, \dots, k$.*

**Theorem 2.** *Let $f$, $g$ be real functions of $n$ variables, $\lim\limits_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = A$ and $\lim\limits_{\boldsymbol{x}\to\boldsymbol{a}} g(\boldsymbol{x}) = B$. Then $\lim\limits_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) \cdot g(\boldsymbol{x}) = AB$. If, moreover, $B \neq 0$, then $\lim\limits_{\boldsymbol{x}\to\boldsymbol{a}} \dfrac{(\boldsymbol{x})}{g(\boldsymbol{x})} = \dfrac{A}{B}$.*

☛ ***Example***. Find the following limits:

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$, b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$, c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$.

Any $(x,y) \neq (0,0)$ can be expressed in polar coordinates as

$$x = r\cos\varphi,\ y = r\sin\varphi,\ r > 0,\ \varphi \in [0, 2\pi).$$

In the case a) we get

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{r\to 0_+} r\cos\varphi\sin^2\varphi = 0\,,$$

since $\left|\cos\varphi\sin^2\varphi\right| \leq 1$.

In the case b) we get

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0_+} \cos\varphi\sin\varphi = \cos\varphi\sin\varphi.$$

The limit does not exist, because it depends on $\varphi$, i.e., on the direction.

The case c) is more complicated. If $x \neq 0$, i.e., $\varphi \neq 0, \pi$, then

$$\frac{xy^2}{x^2 + y^4} \leq \frac{xy^2}{x^2}$$

and the limit is equal to zero as in the case a).

If $x = 0$, the whole expression is equal to zero, too. The limit is therefore equal to zero in all directions.

Nevertheless, for $x = y^2$, the expression is equal to $\frac{1}{2}$, thus the limit is different from zero on this curve.

# Continuity of functions

**Definition 3.** Let $\boldsymbol{f} : D_f \subset \mathbb{R}^n \to \mathbb{R}^k$ be a mapping, let $\boldsymbol{a} \in D_f$. We say that $\boldsymbol{f}$ is **continuous at the point** $\boldsymbol{a}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $f(\boldsymbol{x}) \in U_\varepsilon(\mathbf{A})$ for all $\boldsymbol{x} \in U_\varepsilon(\boldsymbol{a}) \cap D_f$.

**Definition 4.** Let $\boldsymbol{f} : D_f \subset \mathbb{R}^n \to \mathbb{R}^k$ be a mapping, **M** $\subset D_f$. We say that the function $\boldsymbol{f}(\boldsymbol{x})$ **is continuous at the point** $\boldsymbol{a}$ **with respect to the set M** if if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\boldsymbol{f}(\boldsymbol{x}) \in U_\varepsilon(\mathbf{A})$ for all $\boldsymbol{x} \in U_\varepsilon(\boldsymbol{a}) \cap \mathbf{M}$.

***Remark.*** Obviously, $\boldsymbol{f}(x)$ is continuous at $a \in D_f$ if and only if $\boldsymbol{a}$ is an isolated point of $D_f$ or

$$\lim_{x \to a} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{a}).$$

**Definition 5.** Let $\boldsymbol{f} : D_{\boldsymbol{f}} \subset \mathbb{R}^n \to \mathbb{R}^k$ be a mapping, $\mathbf{X} \subset D_{\boldsymbol{f}}$. We say that $\boldsymbol{f}$ is **continuous on the set X** if and only if it is continuous at every point of **X**.

**Theorem 3.** *(Weierstrass) $f : D_f \subset \mathbb{R}^n \to \mathbb{R}$ be continuous on a compact set **X** $\subset D_f$. Then there exist points $\boldsymbol{x}_m$ and $\boldsymbol{x}_M$ in **X** such that $f(\boldsymbol{x}_m) \leq f(\boldsymbol{x})$ and $f(\boldsymbol{x}_M) \geq f(\boldsymbol{x})$for all $\boldsymbol{x} \in$ **X**. I.e., a function continuous on a compact set attains its minimum and maximum on this set.*