EXERCISES 3 – FUNCTIONS

Domain of a function

☛ **1.** Find the domain of the function $f(x) = \sqrt{2 + x - x^2}$.

Solution.

$$2 + x - x^2 \geq 0 \iff (2 - x)(1 + x) \geq 0 \iff x \in \langle -1, 2\rangle \iff D_f = \langle -1, 2\rangle\,.$$

☛ **2.** Find the domain of the function $f(x) = \sqrt{\dfrac{x}{3 - x^2}}$.

Solution.

$$\left(\frac{x}{3 - x^2} \geq 0\right) \wedge \left(3 - x^2 \neq 0\right) \iff \left(\frac{x}{\left(\sqrt{3} - x\right)\left(\sqrt{3} + x\right)} > 0\right) \wedge \left(x \neq \pm\sqrt{3}\right) \iff$$
$$\iff x \in \left(-\infty, -\sqrt{3}\right) \cup \left\langle 0, \sqrt{3}\right) \iff D_f = \left(-\infty, -\sqrt{3}\right) \cup \left\langle 0, \sqrt{3}\right)\,.$$

3. Find the domain of the following function.

a) $f(x) = \ln(x^2 - 3x + 2)$ $[(-\infty, 1) \cup (2, \infty)]$

b) $\ln(x + \sqrt{x^2 + 2})$ $[(-\infty, \infty)]$

c) $f(x) = \ln(\ln 2 - \ln(x^2 - 5x + 8))$ $[(2, 3)]$

d) $f(x) = \ln(x^2 - 9)$ $[(-\infty, -3) \cup (3, \infty)]$

e) $f(x) = \dfrac{5}{x^2 - 9} + \ln(x^3 - x)$ $[(-1, 0) \cup (1, 3) \cup (3, \infty)]$

f) $f(x) = \sqrt{\ln\left(\dfrac{5x - x^2}{4}\right)} + \ln(x^3 - x)$ $[(1, 4]]$

g) $f(x) = \arcsin\dfrac{1 - x}{1 + x}$ $[\,[0, 1]\,]$

h) $f(x) = \arccos\dfrac{x}{x + 1}$

i) $f(x) = \arcsin\dfrac{2 + x}{x}$

j) $f(x) = \arctan\dfrac{\sqrt{1 - x}}{x + 3}$

k) $f(x) = \ln\sin x$ $[\cup_{k\in\mathbb{Z}}(2k\pi, (2k + 1)\pi)\,]$

l) $f(x) = \dfrac{1}{1+\cos x}$ $[\cup_{k\in\mathbb{Z}}((2k-1)\pi,(2k+1)\pi)\,]$

m) $f(x) = \dfrac{1}{\sqrt{\cos(x/3)-\sqrt{3}/2}}$ $[\cup_{k\in\mathbb{Z}}(6k\pi-\pi/2,6k\pi+\pi/2)]$

o) $f(x) = \sqrt{\sin x}$ $[\cup_{k\in\mathbb{Z}}\langle 2k\pi,(2k+1)\pi\rangle\,]$

p) $f(x) = \sqrt{|\sin x|}$ $[\mathbb{R}]$

q) $f(x) = \ln|\sin x|$ $[\,\cup_{k\in\mathbb{Z}}(k\pi,(k+1)\pi)\,]$

r) $f(x) = \dfrac{\cos^2 x+1}{5-\ln x}$ $[(0,e^5)\cup(e^5,\infty)]$

s) $f(x) = \arccos\dfrac{1-2x}{5}$ $[\,[-2,3]\,]$

t) $f(x) = \ln\arcsin\dfrac{1+x}{1-x}$ $[\,(-1,0]\,]$

Inverse functions

4. Find an inverse function to the function

a) $y = e^{x-1}+2$ $[\,y = 1+\ln(x-2)\,,\ x\in(2,\infty)\,]$

b) $y = 10^{x+1}$ $[\,y = \log_{10}(x/10)\,,\ x\in(0,\infty)\,]$

c) $y = \sinh x$ $[\,y = \operatorname{argsinh}x = \ln\left(x+\sqrt{x^2+1}\right)\,]$

d) $y = \cosh x$ $[\,y = \operatorname{argcosh}x = \ln\left(x+\sqrt{x^2-1}\right)\,]$

e) $y = \tanh x$ $[\,y = \operatorname{argtanh}x = \ln\sqrt{\dfrac{1+x}{1-x}}\,]$

f) $y = \coth x$ $[\,y = \operatorname{argcoth}x = \ln\sqrt{\dfrac{x+1}{x-1}}\,]$

g) $y = 5\arccos\sqrt{1-x^2}, \quad x\in[0,1]$ $[\,y = \sin\dfrac{x}{5}\ x\in[0,5\pi/2]\,]$

h) $y = 2\sin 3x$ $[\,y = \dfrac{1}{3}\arcsin\dfrac{x}{2} =]$

Cyclometric functions

5. Find the exact value

a) $\arccos\dfrac{\sqrt{2}}{2}$ $[\pi/4]$

b) $\arccos(-1)$ $[\pi]$

c) $\arctan 0$ $[0]$

d) $\arctan(-1)$ $[-\pi/4]$

e) $\tan(\arctan(-5))$ $[-5]$

f) $\arcsin\left(\sin(\frac{5\pi}{3})\right)$ $[-\pi/3]$

g) $\cos(\arccos(\pi))$ [does not exist]

h) $\tan\left(\arccos(\frac{2}{3})\right)$ $[\sqrt{5}/2]$

i) $\cos\left(\arcsin(-\frac{3}{5})\right)$ $[4/5]$

j) $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$ $[-\pi/3]$

k) $\arctan\left(-\frac{\sqrt{3}}{3}\right)$ $[-\pi/6]$

6. Write the following expression as an algebraic expression in x :

a) $\sin\arccos 3x,\ 0 \le x \le \frac{1}{3}$ $[\sqrt{1-9x^2}]$

Further properties

7. Prove that the following equalities hold for all $x, y \in \mathbb{R}$:

a) $\cosh^2 x - \sinh^x = 1$

b) $\tanh x \cdot \coth x = 1, \quad x \in \mathbb{R} \setminus \{0\}$

c) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

d) $\cosh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

e) $\cot(\arctan x) = \frac{1}{x}$

f) $\tan(\operatorname{arccot} x) = \frac{1}{x}$

8. Sketch the graph of the function $f(x)$.

a) $f(x) = |x+1| + |x-1|$; b) $f(x) = |x| - x$; c) $f(x) = |x-1| - 1$;

d) $f(x) = x^2 + 2 - |2x+1|$; e) $f(x) = x^2 - 2x + 4$; f) $f(x) = -x^2 + 4x - 6$;

g) $f(x) = 1 - \frac{1-x}{1+x}$; h) $f(x) = \frac{3x+2}{2x-3}$; i) $f(x) = \frac{2x-1}{3x+2}$;

j) $f(x) = \left|\dfrac{x+1}{x-1}\right|$; k) $f(x) = \sqrt{-x-2}$; l) $f(x) = \sqrt{4-x^2}$;

m) $f(x) = \ln\dfrac{1}{x-1}$; n) $f(x) = \ln(x-1)$; o) $f(x) = 1 + e^{x-1}$;

p) $f(x) = e^{-|x-1|}$; q) $f(x) = 1 - \cos 2x$; r) $f(x) = |\cot x - 1|$.

9. Decide weather the function $f(x)$ is odd or even.

a) $f(x) = \sqrt[3]{(1-x)^2} + \sqrt[3]{(1+x)^2}$; b) $f(x) = 2^{-x^2}$; c) $f(x) = x + x^2$;

d) $f(x) = \ln(x + \sqrt{1+x^2})$; e) $f(x) = 3x - x^3$; f) $f(x) = \sin x - \cos x$;

g) $f(x) = \dfrac{a^x - 1}{a^x + 1}$; $a > 0$; h) $f(x) = x \cdot \dfrac{a^x - 1}{a^x + 1}$, $a > 0$; i) $f(x) = \dfrac{\sin x}{x}$.

[a) even; b) even; c) neither even nor odd; d) odd; e) odd;
f) neither even nor odd; g) odd; h) even; i) odd.]

10. Find the domain of the function

a) $f(x) = \dfrac{3}{4-x^2} + \ln(x^3 - x)$ $[(-1,0) \cup (1,2) \cup (2,\infty)]$

b) $f(x) = \dfrac{x^2}{1+x}$ $[(-\infty,-1) \cup (-1,\infty)]$

c) $f(x) = \dfrac{x}{\sqrt{x^2-3x+2}}$ $[(-\infty,1) \cup (2,\infty)]$

d) $f(x) = \sqrt{\sin 2x} + \sqrt{\sin 3x}$ $[\cup_{k\in\mathbb{Z}}([2k\pi, 2k\pi + \pi/3] \cup [2k\pi + 4\pi/3, 2k\pi + 3\pi/2]\,]$

e) $f(x) = \sqrt{2 + x - x^2}$ $[\,[-1,2]\,]$

f) $f(x) = \ln(x^2 - 4)$ $[\,(-\infty,-2) \cup (2,\infty)\,]$

g) $f(x) = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1-x}{1+x}}$ $[\,\emptyset\,]$

h) $f(x) = \sqrt{\ln\dfrac{5x - x^2}{4}}$ $[\,[1,4]\,]$

i) $f(x) = \sqrt{\sin x + \dfrac{1}{2}}$ $[\cup_{k\in\mathbb{Z}}[(12k-1)\pi/6, (12k+7)\pi/6]]$

j) $f(x) = \ln(\sqrt{x-4} + \sqrt{6-x})$ $[\,[4,6]\,]$

k) $f(x) = \log_2 \log_3 \log_4 x$ $[\,(4,\infty)\,]$

l) $f(x) = \sqrt{3x - x^3}$ $[(-\infty,-\sqrt{3}] \cup [0,\sqrt{3}]\,]$